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AN ANSWER TO A QUESTION OF DAVID A. ROSE

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Dedicated to Professor Manuel Milla Miranda on the occasion of his 60th birthday

Abstract

In 1984 David.A. Rose [3] asked the following question: When a surjection $f : X \to Y$, is weak openness related to the condition $Cl(f(U)) \subseteq f(Cl(U))$ for each open $U \subseteq X$?. In this note we give an alternative answer to his question.

Key Words and Phrases: Topological space, weak continuity, weak openness.

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1 Introduction and Preliminaries

Weak openness for arbitrary functions between topological spaces where introduced by D.A.Rose [3] as natural dual to the weak continuity of N.Levine [2] and have recently been used in C.W.Baker [1] to obtain a decomposition of openness.

In [4] it is shown that a function $f : X \to Y$ is weakly continuous if and only if $Cl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for each open subset V of Y. D.A.Rose in [3], shows that the dual is not true, i.e., weak openness is independent of the condition $Cl(f(U)) \subseteq f(Cl(U))$ for each open subset U of X. Really ([3], Example 2): Let $\mathbf{R}$ be the usual space of real numbers and let M be the Michael line, i.e. the space M is the set of real numbers with topology $T = \{U \cup H : U$ belongs to the usual topology on $\mathbf{R}$ and $H \subseteq R - \mathbf{Q}$ where $\mathbf{Q}$ is the set of rational real numbers$\}$. Define $1_M : \mathbf{R} \to M$ to be the identity function and let $f = 1_M/\mathbf{Q}$ and $g = 1_M/(\mathbf{R} - \mathbf{Q})$ be restrictions of 1_M. Then $f : \mathbf{Q} \to M$ satisfies the condition: $Cl(f(U)) \subseteq f(Cl(U))$ for each open subset U of $\mathbf{Q}$. But f is not weakly open since $f(\mathbf{Q})$ is not open in M. Further, $g : (\mathbf{R}-\mathbf{Q}) \to M$ is weakly open but since $g(\mathbf{R} - \mathbf{Q})$ is not closed, g does not satisfy the condition: $Cl(g(U)) \subseteq g(Cl(U))$ for each open subset U of $(\mathbf{R} - \mathbf{Q})$.

In this note we present the necessary background information concerning the question given in the abstract of the paper (Theorem 2.2). For more related properties of the discussed problem, the reader may consult the paper written by D.A.Rose [3], where he pose the above mentioned question.

Recall that, A function $f : X \to Y$ from a topological space X into a topological space Y is called weakly open [4] if, $f(U) \subseteq Int(f(Cl(U)))$ for each open set U in X.

2 The Main Result

To state the result precisely, the following theorem is proved.

Theorem 2.1. *Let $f : X \to Y$ be a function from a topological space X into a topological space Y. Then f is weakly open if and only if for each $x \in X$ and each open set U of X containing x, there exists an open set V containing $f(x)$ such that $V \subseteq f(Cl(U))$.*

Proof. Necessity. Let $x \in X$ and U be an open set in X with $x \in U$. Since f is weakly open, $f(x) \in f(U) \subseteq Int(f(Cl(U)))$. Let $V = Int(f(Cl(U)))$. Therefore $f(x) \in V$ and $V \subseteq f(Cl(U))$.

Sufficiency. Let U be an open set in X and let $y \in f(U)$. By hypothesis, $V \subseteq f(Cl(U))$ for some V open in Y containing y. Therefore, we have $y \in V \subseteq Int(f(Cl(U)))$. This shows that $f(U) \subseteq Int(f(Cl(U)))$, i.e., f is a weakly open function. □

Theorem 2.2. *Let $f : X \to Y$ be a bijective function. Then f is weakly open if and only if $Cl(f(U)) \subseteq f(Cl(U))$ for every open set U of X.*

Proof. Necessity. Let U be an open set of X and let $y \notin f(Cl(U))$. Then $f^{-1}(y) \notin Cl(U)$. Therefore there exists an open set G of X containing $f^{-1}(y)$ such that $G \cap U = \emptyset$. Since U is open in X, we have $Cl(G) \cap U = \emptyset$ and since f is injective $Int(f(Cl(G))) \cap f(U) = \emptyset$. Since f is weakly open, $y \in f(G) \subseteq Int(f(Cl(G)))$ where $Int(f(Cl(G))$ is an open subset of Y. Therefore $y \notin Cl(f(U))$. This shows that $Cl(f(U)) \subseteq f(Cl(U))$.

Sufficiency. Let $x \in X$ and U be an open set in X containing x. Since $U \cap (X - Cl(U)) = \emptyset$, we have $x \notin Cl(X - Cl(U))$. Therefore, $f(x) \notin f(Cl(X - Cl(U)))$. Since $X - Cl(U)$ is an open subset in X, then by hypothesis $f(x) \notin Cl(f(X - Cl(U)))$. Thus there exists an open subset V of Y with $f(x) \in V$ such that $V \cap f(X - Cl(U)) = \emptyset$. Therefore, we obtain $f^{-1}(V) \cap (X - Cl(U)) = \emptyset$ and hence and since f is surjective $V \subseteq f(Cl(U))$. By Theorem 2.1, we have that f is weakly open. □

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