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ON A GENERALIZATION OF APPELL'S FUNCTIONS OF TWO VARIABLES

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Abstract

The present paper introduces 10 Appell's type generalized functions $M_i, i = 1, 2, \ldots, 10$ by considering the product of two ${}_3F_2$ functions instead of product of two Gauss functions taken by Appell to define F_1, F_2, F_3 and F_4 functions. In the concluding remark it has been pointed out that by considering the product of two ${}_nF_{n-1}$ functions a set of $n^2 + n - 2$ functions analogus to Appell functions will emerge. The paper contains fractional derivative representations, integral representations, symbolic forms and expansion formulae similar to those obtained by Burchnall and Chaundy for the four Appell's functions, have been obtained for these newly defined functions $M_1, M_2, \ldots, M_{10}$.

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1 Introduction

The great success of the theory of hypergeometric functions of a single variable has stimulated the development of a corresponding theory in two or more variables. In 1880, P. Appell [1] considered the product of two Gauss functions from which the four Appell's functions emerged. Later, in 1893 Lauricella [4] further generalized the four Appell functions F_1, F_2, F_3 and F_4 to functions of n-variables denoted by $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ where $F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} = {}_2F_1$ and $F_A^{(2)} = F_2$, $F_B^{(2)} = F_3$, $F_C^{(2)} = F_4$ and $F_D^{(2)} = F_1$.

During 1940-41, Burchnall and Chaundy [2, 3] obtained a large number of expansions of Appell's double hypergeometric functions. Particularly in [3] they obtained certain interesting integral representations for F_4 and in its last section gave a glimpse of possible extension of their result to functions of higher order (i.e. with more parameters) in two variable for instance they defined.

$$
{}_{p+1}F_p^{(2)}\left[\begin{array}{lll} a: & b_1, \; b_2, \ldots, b_p; \; b_1', b_2', \ldots, b_p'; & \\ & & x, y \\ & c_1, \; c_2, \ldots, c_p; \; c_1', c_2', \ldots, c_p'; & \end{array}\right]
$$

$$
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+m}\,(b_1)_m \cdots (b_p)_m\,(b_1')_n \cdots (b_p')_n}{m!\,n!\,(c_1)_m \cdots (c_p)_m\,(c_1')_n \cdots (c_p')_n}\, x^m\, y^n
$$

$$
= \nabla(a)_{p+1}F_p\left[\begin{array}{ll} a, & b_1, \ldots, b_p; \quad x \\ & c_1, \ldots, c_p; \end{array}\right] {}_{p+1}F_p\left[\begin{array}{ll} a, & b_1', \ldots, b_p'; \quad y \\ & c_1', \ldots, c_p'; \end{array}\right] \tag{1.1}
$$

and gave the result

$$
{}_{p+1}F_p^{(2)}\left[\begin{array}{lll} a: & b_1, \ldots, b_p; \; b_1', \ldots, b_p'; & \\ & & x, y \\ & c_1, \ldots, c_p; \; c_1', \ldots, c_p'; & \end{array}\right]
$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b_1)_r \dots (b_p)_r (b'_1)_r \dots (b'_p)_r \, x^r y^r}{r! (c_1)_r \dots (c_p)_r (c'_1)_r \dots (c'_p)_r}$$

$${}_{p+1}F_p \left[\begin{matrix} a+r, & b_1+r,\dots,b_p+r; & x \\ & c_1+r,\dots,c_p+r; & \end{matrix} \right] {}_{p+1}F_p \left[\begin{matrix} a+r, & b'_1+r,\dots,b'_p+r; & y \\ & c'_1+r,\dots,c'_p+r; & \end{matrix} \right] \quad (1.2)$$

Motivated by this section of [3] and the fact that such functions were encountered during our study of two variable analogues of Saigo's [6] fractional integral operators in a separate communication, we consider in this paper the product of ${}_3F_2$ hypergeometric functions viz.

$${}_3F_2 (a, b, c; d, e; x) \; {}_3F_2(a', b', c'; d', e'; y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n (c)_m (c')_n}{(d)_m (d')_n (e)_m (e')_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.3)$$

This double series, in itself, yields nothing new, but if one or more of the five pairs of products

$$(a)_m (a')_n, \; (b)_m (b')_n, \; (c)_m (c')_n, \; (d)_m (d')_n, \; (e)_m (e')_n$$

be replaced by the corresponding expressions

$$(a)_{m+n}, \; (b)_{m+n}, \; (c)_{m+n}, \; (d)_{m+n}, \; (e)_{m+n}$$

we are lead to eleven distinct possibilities of getting new functions. One such possibility, however, gives us de double series

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n} (c)_{m+n}}{(d)_{m+n} (e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

which is simply the ${}_3F_2$ hypergeometric series for

$${}_3F_2(a,\, b,\, c;\, d,\, e;\, x+y),$$

since it is easily verified that (c.f., e.g. H. M. Srivastava [7])

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m+n) \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} \tag{1.4}$$

The remaining ten possibilities lead to the ten generalized Appell's type functions of two variables, which are as defined below subject to suitable convergence conditions:

$$M_1\,(a,\, a',\, b,\, b',\, c,\, c';\, d,\, e,\, e';\, x,\, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m\,(a')_n\,(b)_m\,(b')_n\,(c)_m\,(c')_n}{(d)_{m+n}\,(e)_m\,(e')_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.5}$$

$$M_2\,(a,\, a',\, b,\, b',\, c,\, c';\, d,\, e;\, x,\, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m\,(a')_n\,(b)_m\,(b')_n\,(c)_m\,(c')_n}{(d)_{m+n}\,(e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.6}$$

$$M_3\,(a,\, b,\, b',\, c,\, c';\, d,\, d',\, e,\, e';\, x,\, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n\,(c)_m\,(c')_n}{(d)_m\,(d')_n\,(e)_m(e')_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.7}$$

$$M_4\,(a,\, b,\, b',\, c,\, c';\, d,\, e,\, e';\, x,\, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n\,(c)_m\,(c')_n}{(d)_{m+n}\,(e)_m(e')_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.8}$$

$$M_5(a, b, b', c, c'; d, e, x, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n (c)_m (c')_n}{(d)_{m+n} (e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.9}$$

$$M_6(a, b, c, c'; d, d', e, e', x, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n} (c)_m (c')_n}{(d)_m (d')_n (e)_m (e')_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.10}$$

$$M_7(a, b, c, c'; d, e, e', x, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n} (c)_m (c')_n}{(d)_{m+n} (e)_m (e')_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.11}$$

$$M_8(a, b, c, c'; d, e, x, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n} (c)_m (c')_n}{(d)_{m+n} (e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.12}$$

$$M_9(a, b, c; d, d', e, e'; x, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n} (c)_{m+n}}{(d)_m (d')_n (e)_m (e')_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.13}$$

$$M_{10}(a, b, c; d, e, e'; x, y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n} (c)_{m+n}}{(d)_{m+n} (e)_m (e')_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.14}$$

2 Symbolic Forms

Burchnall and Chaundy [2, 3] introduced the operators

$$\nabla(h) \equiv \frac{\Gamma(h)\,\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\,\Gamma(\delta'+h)}, \tag{2.1}$$

$$\Delta(h) \equiv \frac{\Gamma(\delta+h)\,\Gamma(\delta'+h)}{\Gamma(h)\,\Gamma(\delta+\delta'+h)}, \tag{2.2}$$

Where $\delta \equiv x\frac{\partial}{\partial x}$ and $\delta' \equiv y\frac{\partial}{\partial y}$

by means of which they factorized the four Appell's functions and obtained certain transformations of these functions. These symbolic forms were used by them to obtain a large member of expansions of Appell's functions in terms of each other, of Appell's functions in terms of products of ordinary hypergeometric functions, or vice versa.

In this section we have followed Burchnall and Chaundy's method to obtain the following factorizations of our newly defined functions M_i, $i = 1, 2, \ldots, 10$.

$$\begin{aligned} M_1 \quad & (a,\, a',\, b,\, b',\, c,\, c';\, d,\, e,\, e';\, x,\, y) \\ = \quad & \Delta(d)\, {}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\, {}_3F_2(a',\, b',\, c';\, d,\, e';\, y) \end{aligned} \tag{2.3}$$

$$\begin{aligned} M_2 \quad & (a,\, a',\, b,\, b',\, c,\, c';\, d,\, e;\, x,\, y) \\ = \quad & \Delta(d)\Delta(e)\, {}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\, {}_3F_2(a',\, b',\, c';\, d,\, e;\, y) \end{aligned} \tag{2.4}$$

$$\begin{aligned} M_3 \quad & (a,\, b,\, b',\, c,\, c';\, d,\, d',\, e,\, e';\, x,\, y) \\ = \quad & \nabla(a)\, {}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\, {}_3F_2(a,\, b',\, c';\, d',\, e';\, y) \end{aligned} \tag{2.5}$$

$$\begin{aligned} M_4 \quad & (a,\, b,\, b',\, c,\, c';\, d,\, e,\, e';\, x,\, y) \\ = \quad & \nabla(a)\,\Delta(d)\, {}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\, {}_3F_2(a,\, b',\, c';\, d,\, e';\, y) \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
M_5 \quad & (a,\, b,\, b',\, c,\, c';\, d,\, e;\, x,\, y) \\
= \quad & \nabla(a)\,\Delta(d)\,\Delta(e)\,{}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\,{}_3F_2(a,\, b',\, c';\, d,\, e;\, y) \quad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
M_6 \quad & (a,\, b,\, c,\, c';\, d,\, d',\, e,\, e';\, x,\, y) \\
= \quad & \nabla(a)\,\nabla(b)\,{}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\,{}_3F_2(a,\, b,\, c';\, d',\, e';\, y) \quad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
M_7 \quad & (a,\, b,\, c,\, c';\, d,\, e,\, e';\, x,\, y) \\
= \quad & \nabla(a)\,\nabla(b)\,\Delta(d)\,{}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\,{}_3F_2(a,\, b,\, c';\, d,\, e';\, y) \quad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
M_8 \quad & (a,\, b,\, c,\, c';\, d,\, e;\, x,\, y) \\
= \quad & \nabla(a)\,\nabla(b)\,\Delta(d)\,\Delta(e)\,{}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\,{}_3F_2(a,\, b,\, c';\, d,\, e;\, y) \\
& \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
M_9 \quad & (a,\, b,\, c;\, d,\, d',\, e,\, e';\, x,\, y) \\
= \quad & \nabla(a)\,\nabla(b)\,\nabla(c)\,{}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\,{}_3F_2(a,\, b,\, c;\, d',\, e';\, y) \quad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
M_{10} \quad & (a,\, b,\, c;\, d,\, e,\, e';\, x,\, y) \\
= \quad & \nabla(a)\,\nabla(b)\,\nabla(c)\,\Delta(d)\,{}_3F_2(a,\, b,\, c;\, d,\, e;\, x)\,{}_3F_2(a,\, b,\, c;\, d,\, e';\, y) \\
& \qquad (2.12)
\end{aligned}
$$

$$
M_4 \left[\begin{matrix} a, b, & b', c, c'; & x, y \\ & d, e, e'; & \end{matrix} \right] = \nabla(a) M_1 \left[\begin{matrix} a, a, b, & b', c, c'; & x, y \\ & d, e, e'; & \end{matrix} \right] \quad (2.13)
$$

$$
M_1 \left[\begin{matrix} a, a, b, & b', c, c'; & x, y \\ & d, e, e'; & \end{matrix} \right] = \Delta(a) M_4 \left[\begin{matrix} a, b, & b', c, c'; & x, y \\ & d, e, e'; & \end{matrix} \right] \quad (2.14)
$$

$$
M_5 \left[\begin{matrix} a, b, b', & c, c'; & x, y \\ & d, e; & \end{matrix} \right] = \nabla(a) M_2 \left[\begin{matrix} a, a, b, b', & c, c'; & x, y \\ & d, e; & \end{matrix} \right] \quad (2.15)
$$

$$M_2 \left[\begin{matrix} a,a,b,b', & c,c'; & x,y \\ & d,e; & \end{matrix}\right] = \Delta(a) M_5 \left[\begin{matrix} a,b,b', & c,c'; & x,y \\ & d,e; & \end{matrix}\right] \tag{2.16}$$

$$M_4 \left[\begin{matrix} a,b, & b',c,c'; & x,y \\ & d,e,e'; & \end{matrix}\right] = \Delta(d) M_3 \left[\begin{matrix} a,b, & b',c,c'; & x,y \\ & d,d,e,e'; & \end{matrix}\right] \tag{2.17}$$

$$M_3 \left[\begin{matrix} a,b,b',c,c'; & x,y \\ d,d,e,e'; & \end{matrix}\right] = \nabla(d) M_4 \left[\begin{matrix} a,b, & b',c,c'; & x,y \\ & d,e,e'; & \end{matrix}\right] \tag{2.18}$$

$$ {}_3F_2 \left[\begin{matrix} a,b,c; & x+y \\ d,e; & \end{matrix}\right] = \nabla(c) M_8 \left[\begin{matrix} a,b, & c,c; & x,y \\ & d,e; & \end{matrix}\right] \tag{2.19}$$

$$ {}_3F_2 \left[\begin{matrix} a,b,c; & x+y \\ d,e; & \end{matrix}\right] = \Delta(e) M_{10} \left[\begin{matrix} a,b,c; & x,y \\ d,e,e'; & \end{matrix}\right] \tag{2.20}$$

$$ {}_3F_2 \left[\begin{matrix} a,b,c; & x+y \\ d,e; & \end{matrix}\right] = \nabla(c)\Delta(e) M_7 \left[\begin{matrix} a,b,c,c; & x,y \\ d,e,e; & \end{matrix}\right] \tag{2.21}$$

$$M_6 \left[\begin{matrix} a,b,c,c'; & x,y \\ d,d',e,e'; & \end{matrix}\right] = \nabla(b) M_3 \left[\begin{matrix} a,b,b,c,c'; & x,y \\ d,d',e,e'; & \end{matrix}\right] \tag{2.22}$$

$$M_3 \left[\begin{matrix} a,b,b,c,c'; & x,y \\ d,d',e,e'; & \end{matrix}\right] = \Delta(b) M_6 \left[\begin{matrix} a,b,c,c'; & x,y \\ d,d',e,e'; & \end{matrix}\right] \tag{2.23}$$

$$M_7 \left[\begin{matrix} a,b,c,c'; & x,y \\ d,e,e'; & \end{matrix}\right] = \nabla(b) M_4 \left[\begin{matrix} a,b, & b,c,c'; & x,y \\ & d,e,e'; & \end{matrix}\right] \tag{2.24}$$

$$M_4 \quad \left[\begin{matrix} a,b,b,c,c'; & x,y \\ d,e,e'; & \end{matrix} \right] = \Delta(b) M_7 \left[\begin{matrix} a,b,c,c'; & x,y \\ d,e,e'; & \end{matrix} \right] \tag{2.25}$$

$$M_8 \quad \left[\begin{matrix} a,b, & c,c'; & x,y \\ & d,e; & \end{matrix} \right] = \nabla(b) M_5 \left[\begin{matrix} a,b,b, & c,c'; & x,y \\ & d,e; & \end{matrix} \right] \tag{2.26}$$

$$M_5 \quad \left[\begin{matrix} a,b,b, & c,c'; & x,y \\ & d,e; & \end{matrix} \right] = \Delta(b) M_8 \left[\begin{matrix} a,b, & c,c'; & x,y \\ & d,e; & \end{matrix} \right] \tag{2.27}$$

$$M_9 \quad \left[\begin{matrix} & a,b,c; & x,y \\ d, & d',e,e'; & \end{matrix} \right] = \nabla(c) M_6 \left[\begin{matrix} a,b,c,c; & x,y \\ d,d',e,e'; & \end{matrix} \right] \tag{2.28}$$

$$M_6 \quad \left[\begin{matrix} a,b,b,c,c; & x,y \\ d,d',e,e'; & \end{matrix} \right] = \Delta(c) M_9 \left[\begin{matrix} a,b,c; & x,y \\ d,d',e,e'; & \end{matrix} \right] \tag{2.29}$$

$$M_{10} \quad \left[\begin{matrix} a,b,c; & x,y \\ d,e,e'; & \end{matrix} \right] = \nabla(c) M_7 \left[\begin{matrix} a,b,c,c; & x,y \\ d,e,e'; & \end{matrix} \right] \tag{2.30}$$

$$M_7 \quad \left[\begin{matrix} a,b,c,c; & x,y \\ d,e,e'; & \end{matrix} \right] = \Delta(c) M_{10} \left[\begin{matrix} a,b,c; & x,y \\ d,e,e'; & \end{matrix} \right] \tag{2.31}$$

3 Fractional Derivative Representation

In 1731 Euler extended the derivative formula

$$\begin{aligned} D_z^n\{z^\lambda\} &= \lambda(\lambda-1)\dots(\lambda-n+1)z^{\lambda-n} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} z^{\lambda-n} (n=0,1,2,\dots) \end{aligned} \tag{3.1}$$

to the general form:

$$D_z^{\mu}\{z^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} z^{\lambda-\mu} \tag{3.2}$$

where μ is an arbitrary complex number.

Using (3.2) we give here the following fractional derivative representations:

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}\, {}_2F_1\left[\begin{matrix} \alpha, & \beta; & az \\ & \gamma; & \end{matrix}\right]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1}\, {}_3F_2\left[\begin{matrix} \lambda, & \alpha,\beta; & az \\ & \mu,\gamma; & \end{matrix}\right] \tag{3.3}$$

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}\, {}_2F_1\left[\begin{matrix} \alpha, & \beta; & xz \\ & \gamma; & \end{matrix}\right] {}_2F_1\left[\begin{matrix} \alpha', & \beta'; & yz \\ & \gamma'; & \end{matrix}\right]\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} M_4(\lambda,\alpha,\alpha',\beta,\beta';\mu,\gamma,\gamma';xz,yz) \tag{3.4}$$

$$D_y^{b'-d'} D_z^{c'-e'}\left\{y^{b'-1} z^{c'-1} (1-yz)^{-a}\, {}_3F_2\left[\begin{matrix} a, & b,c; & \frac{x}{1-yz} \\ & d,e; & \end{matrix}\right]\right\}$$

$$= \frac{\Gamma(b')\Gamma(c')}{\Gamma(d')\Gamma(e')} y^{d'-1} z^{e'-1} M_3(a,b,b',c,c';d,d',e,e';x,yz) \tag{3.5}$$

$$D_x^{c-e} D_y^{c'-e'}\{x^{c-1} y^{c'-1}\, F_2[a;b,b';d,d';x,y]\}$$

$$= \frac{\Gamma(c)\Gamma(c')}{\Gamma(e)\Gamma(e')} x^{e-1} y^{e'-1} M_3(a,b,b',c,c';d,d',e,e';x,y) \tag{3.6}$$

$$D_x^{c-e} D_y^{c'-e'}\{x^{c-1} y^{c'-1}\, F_1[a;b,b';d;x,y]\}$$

$$= \frac{\Gamma(c)\Gamma(c')}{\Gamma(e)\Gamma(e')} x^{e-1} y^{e'-1} M_4(a,b,b',c,c';d,e,e';x,y) \tag{3.7}$$

$$D_x^{c-e} D_y^{c'-e'} \{x^{c-1} y^{c'-1} F_3[a, a'; b, b'; d; x, y]\}$$

$$= \frac{\Gamma(c)\Gamma(c')}{\Gamma(e)\Gamma(e')} x^{e-1} y^{e'-1} M_1(a, a', b, b', c, c'; d, e, e'; x, y) \tag{3.8}$$

$$D_x^{c-e} D_y^{c'-e'} \{x^{c-1} y^{c'-1} F_4[a; b; d, d'; x, y]\}$$

$$= \frac{\Gamma(c)\Gamma(c')}{\Gamma(e)\Gamma(e')} x^{e-1} y^{e'-1} M_6(a, b, c, c'; d, d', e, e'; x, y) \tag{3.9}$$

$$D_z^{a-d} \left\{ z^{a-1} F_2 \left[\begin{matrix} b, & c, c'; & xz, yz \\ & e, e'; & \end{matrix} \right] \right\}$$

$$= \frac{\Gamma(a)}{\Gamma(d)} z^{d-1} M_7 \left[\begin{matrix} a, & b, c, c'; & xz, yz \\ & d, e, e'; & \end{matrix} \right] \tag{3.10}$$

$$D_z^{a-d} \left\{ z^{a-1} F_1 \left[\begin{matrix} b, & c, c'; & xz, yz \\ & e; & \end{matrix} \right] \right\}$$

$$= \frac{\Gamma(a)}{\Gamma(d)} z^{d-1} M_8 \left[\begin{matrix} a, b, & c, c'; & xz, yz \\ & d, e; & \end{matrix} \right] \tag{3.11}$$

$$D_z^{a-d} \left\{ z^{a-1} F_3 \left[\begin{matrix} b, b', c, & c'; & xz, yz \\ & e; & \end{matrix} \right] \right\}$$

$$= \frac{\Gamma(a)}{\Gamma(d)} z^{d-1} M_5 \left[\begin{matrix} a, b, b', & c, c'; & xz, yz \\ & d, e; & \end{matrix} \right] \tag{3.12}$$

$$D_z^{a-d} \left\{ z^{a-1} F_4 \left[\begin{matrix} b, c; & xz, yz \\ e, e'; & \end{matrix} \right] \right\}$$

$$= \frac{\Gamma(a)}{\Gamma(d)} z^{d-1} M_{10} \left[\begin{matrix} a, b, c; & xz, yz \\ d, e, e'; & \end{matrix} \right] \tag{3.13}$$

$$D_z^{a-d}\left\{z^{a-1}F_3\left[\begin{array}{ccc} a,a',b, & b',c,c'; & xz,yz \\ & a,e,e'; & \end{array}\right]\right\}$$

$$= \frac{\Gamma(a)}{\Gamma(d)} z^{d-1} M_1 \left[\begin{array}{ccc} a,a',b, & b',c,c'; & xz,yz \\ & d,e,e'; & \end{array}\right] \tag{3.14}$$

$$D_z^{a-d}\, D_w^{d-d'}\left\{w^{d-1}z^{a-1}\nabla(d)F_4\left[\begin{array}{cc} b,c; & xz,wyz \\ e,e'; & \end{array}\right]\right\}$$

$$= \frac{\Gamma(a)}{\Gamma(d')} w^{d'-1} z^{d-1} M_9 \left[\begin{array}{ccc} & a,b,c; & xz,wyz \\ d, & d',e,e'; & \end{array}\right] \tag{3.15}$$

$$D_z^{a-d} D_w^{a'-a}\left\{w^{a'-1}z^{a-1}\Delta(a)F_3\left[\begin{array}{ccc} b,b',c, & c'; & xz,wyz \\ & e; & \end{array}\right]\right\}$$

$$= \frac{\Gamma(a')}{\Gamma(d)} w^{a-1} z^{d-1} M_2 \left[\begin{array}{ccc} a,a',b,b', & c,c'; & xz,wyz \\ & d,e; & \end{array}\right] \tag{3.16}$$

4 Integral Representations

In the theory of Eulerian integrals, the elementary formulas

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, R1(\alpha) > 0,\ R1(\beta) > 0 \tag{4.1}$$

and

$$\int\int u^{\alpha-1}v^{\beta-1}(1-u-v)^{\gamma-1}dudv = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)},$$

$$u \geq 0, v \geq, u+v \leq 1, R1(\alpha) > 0, R1(\beta) > 0, R1(\gamma) > 0 \tag{4.2}$$

are fairly well known. Making use of (4.1) and (4.2) we give the following integral representations for $M_1, M_2, \ldots, M_{10}$.

$$\frac{\Gamma(c)\Gamma(c')\Gamma(d-c-c')}{\Gamma(d)} M_1(a,a',b,b',c,c';d,e,e';x,y)$$

$$= \int\int u^{c-1}v^{c'-1}(1-u-v)^{d-c-c'-1} {}_2F_1\left[\begin{array}{c} a,b;ux \\ e; \end{array}\right] {}_2F_1\left[\begin{array}{c} a',b';uy \\ e'; \end{array}\right] dudv, \tag{4.3}$$

$u \geq 0$, $v \geq 0$, $u+v \leq 1$, $Rl(c) > 0$, $Rl(c') > 0$,

$Rl(d-c-c') > 0$,

$$\frac{\Gamma(c)\Gamma(c')}{\Gamma(c+c')} M_1\left[\begin{array}{ccc} a,a', & b,b',c,c'; & x,y \\ & c+c',d,d'; & \end{array}\right]$$

$$= \int_0^1 u^{c-1}(1-u)^{c'-1} {}_2F_1\left[\begin{array}{c} a,b;ux \\ d; \end{array}\right] {}_2F_1\left[\begin{array}{c} a',b';(1-u)y \\ d'; \end{array}\right] du \tag{4.4}$$

$Rl(c) > 0$, $Rl(c') > 0$,

$$\frac{\Gamma(c)\Gamma(c')\Gamma(d-c-c')}{\Gamma(d)} M_2\left[\begin{array}{ccc} a,a',b,b', & c,c'; & x,y \\ & d,e; & \end{array}\right]$$

$$= \int\int u^{c-1}v^{c'-1}(1-u-v)^{d-c-c'-1} F_3\left[\begin{array}{c} a,a',b;b';ux,vy \\ e; \end{array}\right] dudv \tag{4.5}$$

$u \geq 0$, $v \geq 0$, $u+v \leq 1$, $Rl(c) > 0$, $Rl(c') > 0$,

$Rl(d-c-c') > 0$,

$$\frac{\Gamma(c)\Gamma(c')}{\Gamma(c+c')} M_2\left[\begin{array}{ccc} a,a',b, & b',c,c'; & x,y \\ & c+c',d; & \end{array}\right]$$

$$= \int_0^1 u^{c-1}(1-u)^{c'-1} F_3\left[\begin{array}{ccc} a,a',b, & b'; & ux,(1-u)y \\ & d; & \end{array}\right] du \tag{4.6}$$

$Rl(c) > 0$, $Rl(c') > 0$,

$$\frac{\Gamma(b)\Gamma(b')\Gamma(c)\Gamma(c')\Gamma(d-b)\Gamma(d'-b')\Gamma(e-c)\Gamma(e'-c')}{\Gamma(d)\Gamma(d')\Gamma(e)\Gamma(e')}$$

$$\times \quad M_3\left[\begin{array}{lll} a, & b,b',c,c'; & x,y \\ & d,d',e,e'; & \end{array}\right]$$

$$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 t^{b-1}\, u^{b'-1} v^{c-1} w^{c'-1} (1-t)^{d-b-1}(1-u)^{d'-b'-1}$$

$$(1-v)^{e-c-1}(1-w)^{e'-c'-1}(1-tvx-uwy)^{-a}\; dw\; dv\; dudt \qquad (4.7)$$

$$Rl(d) > Rl(b) > 0, Rl(d') > Rl(b') > 0, Rl(e) > Rl(c) > 0,$$
$$Rl(e') > Rl(c') > 0,$$

$$\frac{\Gamma(c)\Gamma(c')\Gamma(d-c-c')}{\Gamma(d)} M_4\left[\begin{array}{lll} a,b, & b',c,c'; & x,y \\ & d,e,e'; & \end{array}\right]$$

$$= \int\int u^{c-1}v^{c'-1}(1-u-v)^{d-c-c'-1} F_2\left[\begin{array}{ll} a,b,b'; & ux,vy \\ e,e'; & \end{array}\right] du\, dv \qquad (4.8)$$

$u \geq 0,\ v \geq 0,\ u+v \leq 1,\ Rl(c) > 0,\ Rl(c') > 0,$

$Rl(d-c-c') > 0,$

$$\frac{\Gamma(c)\Gamma(c')\Gamma(d-c-c')}{\Gamma(d)} M_5\left[\begin{array}{lll} a,b,b', & c,c'; & x,y \\ & d,e'; & \end{array}\right]$$

$$= \int\int u^{c-1}v^{c'-1}(1-u-v)^{d-c-c'-1} F_1\left[\begin{array}{ll} a,b, & b';ux,vy \\ & e; \end{array}\right] du\, dv \qquad (4.9)$$

$u \geq 0,\ v \geq 0,\ u+v \leq 1,\ Rl(c) > 0,\ Rl(c') > 0,$

$Rl(d-c-c') > 0,$

$$\frac{\Gamma(c)\Gamma(c')\Gamma(d-c)\Gamma(d'-c')}{\Gamma(d)\Gamma(d')} M_6 \left[\begin{array}{ll} a,b,c,c'; & x,y \\ d,d',e,e'; & \end{array} \right]$$

$$= \int_0^1 \int_0^1 u^{c-1} v^{c'-1} (1-u)^{d-c-1} (1-v)^{d'-c'-1} F_4 \left[\begin{array}{ll} a,b; & ux,vy \\ e,e'; & \end{array} \right] du\, dv \qquad (4.10)$$

$Rl(d) > Rl(c) > 0, Rl(d') > Rl(c') > 0,$

$$\frac{\Gamma(c)\Gamma(c')\Gamma(d-c-c')}{\Gamma(d)} M_7 \left[\begin{array}{lll} a, & b,c,c'; & x,y \\ & d,e,e'; & \end{array} \right]$$

$$= \int \int u^{c-1} v^{c'-1} (1-u-v)^{d-c-c'-1} F_4 \left[\begin{array}{ll} a,b; & ux,vy \\ e,e'; & \end{array} \right] du\, dv \qquad (4.11)$$

$u \geq 0,\ v \geq 0,\ u+v \leq 1,\ Rl(c) > 0,\ Rl(c') > 0,$

$Rl(d-c-c') > 0,$

$$\frac{\Gamma(c)\Gamma(c')\Gamma(d-c-c')}{\Gamma(d)} M_8 \left[\begin{array}{lll} a,b, & c,c'; & x,y \\ & d,e; & \end{array} \right]$$

$$= \int \int u^{c-1} v^{c'-1} (1-u-v)^{d-c-c'-1} {}_2F_1 \left[\begin{array}{ll} a,b; & ux+vy \\ e'; & \end{array} \right] du\, dv \qquad (4.12)$$

$u \geq 0,\ v \geq 0,\ u+v \leq 1,\ Rl(c) > 0,\ Rl(c') > 0,$

$Rl(d-c-c') > 0,$

$$\frac{\{\Gamma(c)\}2\Gamma(d-c)\Gamma(d'-c')\Gamma(c'-c)}{\Gamma(d)\Gamma(d')} M_9 \left[\begin{array}{lll} & a,b,c; & x,y \\ d, & d',e,e'; & \end{array} \right]$$

$$= \int_0^1 \int_0^1 \int_0^1 u^{c-1} v^{c'-1} w^{c-1} (1-u)^{d-c-1} (1-v)^{d'-c'-1}$$

$$(1-w)^{c'-c-1} \nabla(c) F_4 \left[\begin{array}{ll} a,b; & ux,vwy \\ e,e'; & \end{array} \right] du\, dv,\ dw \qquad (4.13)$$

$Rl(d) > Rl(c) > 0, Rl(d') > Rl(c') > 0, Rl(c') > Rl(c) > 0,$

$$\frac{\Gamma(c)\Gamma(d-c')}{\Gamma(d)} M_{10}\left[\begin{array}{ll} a,b,c; & x,y \\ d,e,e'; & \end{array}\right]$$

$$= \int_0^1 u^{c-1}(1-u)^{d-c-1}\, F_4\left[\begin{array}{ll} a,b; & ux,vy \\ e,e'; & \end{array}\right] du \tag{4.14}$$

$Rl(d) > Rl(c) > 0.$

5 Expansions

Following the procedure adopted by Burchnall and Chaundy [2, 3] we obtain the following expansions of $M_i, i = 1, 2, \ldots, 10$:

$$M_1\left[\begin{array}{lll} a,a',b, & b',c,c'; & x,y \\ & d,e,e', & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_r (a')_r (b)_r (b')_r (c)_r (c')_r\, x^r y^r}{r!(d+r-1)_r (d)_{2r} (e)_r (e')_r} \tag{5.1}$$

$${}_3F_2\left[\begin{array}{l} a+r, b+r, c+r; x \\ d+2r, e+r; \end{array}\right] {}_3F_2\left[\begin{array}{l} a'+r, b'+r, c'+r; y \\ d+2r, e'+r; \end{array}\right],$$

$${}_3F_2\left[\begin{array}{lll} a, & b,c, & x \\ & d,e; & \end{array}\right] {}_3F_2\left[\begin{array}{lll} a', & b',c'; & y \\ & d,e; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(a)_r (a')_r (b)_r (b')_r (c)_r (c')_r\, x^r y^r}{r!(d)_r (d)_{2r} (e)_r (e')_r} \tag{5.2}$$

$$M_1\left[\begin{array}{lll} a+r, a'+r, b+r, & b'+r, c+r, c'+r; & x,y \\ & d+2r, e+r, e'+r; & \end{array}\right].$$

Similarly, (2.13) and (2.14) give

$$M_4\left[\begin{array}{ccc} a,b, & b',c,c'; & x,y \\ & d,e,e'; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(a)_r(b)_r(b')_r(c)_r(c')_r\, x^r y^r}{r!(d)_{2r}(e)_r(e')_r} \tag{5.3}$$

$$M_1\left[\begin{array}{ccc} a+r,a'+r,b+r, & b'+r,c+r,c'+r; & x,y \\ & d+2r,e+r,e'+r; & \end{array}\right].$$

and

$$M_1\left[\begin{array}{ccc} a,a,b, & b',c,c'; & x,y \\ & d,e,e'; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(-1)^r(a)_r(b)_r(b')_r(c)_r(c')_r\, x^r y^r}{r!(d)_{2r}(e)_r(e')_r} M_4\left[\begin{array}{c} a,b,b',c,c';x,y \\ d,e,e'; \end{array}\right]. \tag{5.4}$$

Now (2.15) and (2.16) respectively yield

$$M_5\left[\begin{array}{ccc} a,b,b', & c,c'; & x,y \\ & d,e; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(a)_r(b)_r(b')_r(c)_r(c')_r\, x^r y^r}{r!(d)_{2r}(e)_{2r}} \tag{5.5}$$

$$M_2\left[\begin{array}{ccc} a+r,a+r,b+r,b'+r, & c+r,c'+r; & x,y \\ & d+2r,e+2r; & \end{array}\right]$$

and

$$M_2\left[\begin{array}{ccc} a,a,b,b', & c,c'; & x,y \\ & d,e; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_r (b)_r (b')_r (c)_r (c')_r \, x^r y^r}{r!(d)_{2r}(e)_{2r}} \tag{5.6}$$

$$M_5\left[\begin{array}{ccc} a+r,b+r,b'+r, & c+r,c'+r; & x,y \\ & d+2r,e+2r; & \end{array}\right].$$

Further (2.5) gives

$$M_3\left[\begin{array}{ccc} a, & b,b',c,c'; & x,y \\ & d,d',e,e'; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(a)_r (b)_r (b')_r (c)_r (c')_r \, x^r y^r}{r!(d)_r (d')_r (e)_r (e')_r} \tag{5.7}$$

$${}_3F_2\left[\begin{array}{c} a+r,\ b+r,c+r,\ x \\ d+r,e+r; \end{array}\right] {}_3F_2\left[\begin{array}{c} a+r,\ b'+r,c'+r;\ y \\ d+r',e'+r; \end{array}\right]$$

Conversely, we have

$${}_3F_2\left[\begin{array}{ccc} a, & b,c, & x \\ & d,e; & \end{array}\right] {}_3F_2\left[\begin{array}{ccc} a', & b',c'; & y \\ & d',e'; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_r (b)_r (b')_r (c)_r (c')_r \, x^r y^r}{r!(d)_r (d')_r (e)_r (e')_r} M_3\left[\begin{array}{c} a+r, b+r,b'+r,c+r,c'+r; x,y \\ d+r,d'+r,e+r,e'+r; \end{array}\right]. \tag{5.8}$$

From (2.17) and (2.18) we get respectively

$$M_4\left[\begin{array}{ccc} a,b, & b',c,c'; & x,y \\ & d,e,e'; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_{2r} (b)_r (b')_r (c)_r (c')_r \, x^r y^r}{r!(d+r-1)_r (d)_{2r} (e)_r (e')_r} M_3\left[\begin{array}{c} a+2r, b+r,b'+r,c+r,c'+r\ x,y \\ d+2r,d+2r,e+r,e'+r; \end{array}\right] \tag{5.9}$$

and

$$M_3\left[\begin{array}{lll} a, & b, b', c, c'; & x, y \\ & d, d, e, e'; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(a)_{2r}(b)_r(b')_r(c)_r(c')_r\, x^r y^r}{r!(d)_r(d)_{2r}(e)_r(e')_r} M_4\left[\begin{array}{ll} a+2r, b+r, b'+r, c+r, c'+r & x, y \\ d+2r, e+r, e'+r; & \end{array}\right] \tag{5.10}$$

$$M_4\left[\begin{array}{lll} a, b, & b', c, c'; & x, y \\ & d, e, e'; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(a)_r(b)_r(b')_r(c)_r(c')_r(d-a)_r\, x^r y^r}{r!(d+r-1)_r(d)_{2r}(e)_r(e')_r} \tag{5.11}$$

$${}_3F_2\left[\begin{array}{l} a+r,\, b+r, c+r,\, x \\ d+2r, e+r; \end{array}\right] {}_3F_2\left[\begin{array}{l} a+r,\, b'+r, c'+r;\, y \\ d+2r, e'+r; \end{array}\right],$$

$${}_3F_2\left[\begin{array}{lll} a, & b, c, & x \\ & d, e; & \end{array}\right] {}_3F_2\left[\begin{array}{lll} a, & b', c'; & y \\ & d, e'; & \end{array}\right]$$

$$= \sum_{r=0}^{\alpha} \frac{(-1)^r(a)_r(b)_r(b')_r(c)_r(c')_r(d-a)_r\, x^r y^r}{r!(d)_r(d)_{2r}(e)_r(e')_r} \tag{5.12}$$

$$M_4\left[\begin{array}{lll} a+r, b+r, & b'+r, c+r, c'+r; & x, y \\ & d+2r, e+r, e'+r; & \end{array}\right].$$

Now (2.19), (2.20) and (2.21) give respectively

$${}_3F_2\left[\begin{array}{lll} a, & b, c, & x+y \\ & d, e; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(a)_{2r}(b)_{2r}(c)_r\, x^r y^r}{r!(d)_{2r}(e)_{2r}} \tag{5.13}$$

$$M_8\left[\begin{array}{lll} a+2r, b+2r, & c+r, c+r; & x, y \\ & d+2r, e+2r; & \end{array}\right],$$

$$
{}_3F_2\left[\begin{array}{ccc} a, & b, c, & x+y \\ & d, e; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_{2r} (b)_{2r} (c)_{2r}\, x^r y^r}{r!(e+r-1)_r (e)_{2r} (d)_{2r}}
$$

$$
M_{10}\left[\begin{array}{cc} a+2r, b+2r, c+2r; & x, y \\ d+2r, e+2r, e+2r; & \end{array}\right], \tag{5.14}
$$

$$
{}_3F_2\left[\begin{array}{ccc} a, & b, c, & x+y \\ & d, e; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(e-c)_r (a)_{2r} (b)_{2r} (c)_r\, x^r y^r}{r!(e+r-1)_r (d)_{2r} (e)_{2r}}
$$

$$
M_7\left[\begin{array}{ccc} a+2r, & b+2r, c+r, c+r; & x, y \\ & d+2r, e+2r, e+2r; & \end{array}\right]. \tag{5.15}
$$

Now from (2.22 - 2.31), we obtain respectively,

$$
M_6\left[\begin{array}{cc} a, b, c, c'; & x, y \\ d, d', e, e'; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(a)_{2r} (b)_r (c)_r (c')_r\, x^r y^r}{r!(d)_r (d')_r (e)_r (e')_r}
$$

$$
M_3\left[\begin{array}{ccc} a+2r, & b+r, b+r, c+r, c'+r; & x, y \\ & d+r, d'+r, e+r, e'+r; & \end{array}\right], \tag{5.16}
$$

$$
M_3\left[\begin{array}{ccc} a, & b, b, c, c'; & x, y \\ & d, d', e, e'; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_{2r} (b)_r (c)_r (c')_r\, x^r y^r}{r!(d)_r (d')_r (e)_r (e')_r}
$$

$$
M_6\left[\begin{array}{cc} a+2r, b+r, c+r, c'+r; & x, y \\ d+r, d'+r, e+r, e'+r; & \end{array}\right], \tag{5.17}
$$

$$
M_7\left[\begin{array}{ccc} a, & b, c, c'; & x, y \\ & d, e, e'; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(a)_{2r} (b)_r (c)_r (c')_r\, x^r y^r}{r!(d)_{2r} (e)_r (e')_r}
$$

$$
M_4\left[\begin{array}{ccc} a+2r, b+r, & b+r, c+r, c'+r; & x, y \\ & d+2r, e+r, e'+r; & \end{array}\right], \tag{5.18}
$$

$$M_4\left[\begin{array}{ccc} a,b & b,c,c'; & x,y \\ & d,e,e'; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_{2r}(b)_r(c)_r(c')_r\, x^r y^r}{r!(d)_{2r}(e)_r(e')_r}$$

$$M_7\left[\begin{array}{ccc} a+2r, & b+r,c+r,c'+r; & x,y \\ & d+2r,e+r,e'+r; & \end{array}\right], \tag{5.19}$$

$$M_8\left[\begin{array}{ccc} a,b, & c,c'; & x,y \\ & d,e; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(a)_{2r}(b)_r(c)_r(c')_r x^r y^r}{r!(d)_{2r}(e)_{2r}}$$

$$M_5\left[\begin{array}{ccc} a+2r,b+r,b+r, & c+r,c'+r; & x,y \\ & d+2r,e+2r; & \end{array}\right], \tag{5.20}$$

$$M_5\left[\begin{array}{ccc} a,b,b, & c,c'; & x,y \\ & d,e; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_{2r}(b)_r(c)_r(c')_r\, x^r y^r}{r!(d)_{2r}(e)_{2r}}$$

$$M_8\left[\begin{array}{ccc} a+2r,b+r, & c+r,c'+r; & x,y \\ & d+r,e+2r; & \end{array}\right], \tag{5.21}$$

$$M_9\left[\begin{array}{ccc} & a,b,c; & x,y \\ d, & d',e,e'; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(a)_{2r}(b)_{2r}(c)_r\, x^r y^r}{r!(d)_r(d')_r(e)_r(e')_r}$$

$$M_6\left[\begin{array}{cc} a+2r,b+2r,c+r,c+r; & x,y \\ d+r,d'+r,e+r,e'+r; & \end{array}\right], \tag{5.22}$$

$$M_6\left[\begin{array}{cc} a,b,c,c; & x,y \\ d,d',e,e'; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(-1)^r (a)_{2r}(b)_{2r}(c)_r\, x^r y^r}{r!(d)_r(d')_r(e)_r(e')_r}$$

$$M_9\left[\begin{array}{ccc} & a+2r,b+2r,c+r; & x,y \\ d+r, & d'+r,e+r,e'+r; & \end{array}\right], \tag{5.23}$$

$$M_{10}\left[\begin{array}{ll} a,b,c; & x,y \\ d,e,e'; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(a)_{2r}(b)_{2r}(c)_r\, x^r y^r}{r!(d)_{2r}(e)_r(e')_r}$$

$$M_7\left[\begin{array}{lll} a+2r, & b+2r, c+r, c+r; & x,y \\ & d+2r, e+r, e'+r; & \end{array}\right], \tag{5.24}$$

$$M_7\left[\begin{array}{lll} a, & b,c,c; & x,y \\ & d,e,e'; & \end{array}\right] = \sum_{r=0}^{\alpha} \frac{(-1)^r(a)_{2r}(b)_{2r}(c)_r\, x^r y^r}{r!(d)_{2r}(e)_r(e')_r}$$

$$M_{10}\left[\begin{array}{ll} a+2r, b+2r, c+r; & x,y \\ d+2r, e+r, e'+r; & \end{array}\right]. \tag{5.25}$$

6 Concluding Remark

In a manner similar to one considered in this paper if we consider the product of two ${}_4F_3$ functions we get 18 Appell type functions and by considering product of two ${}_5F_4$ functions we get 28 Appell type functions. Thus in general if we consider the product of two ${}_nF_{n-1}$ functions we will get n^2+n-2 Appell type functions. Our subsequent paper's study will be based on this fact.

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