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REORDERINGS OF SERIES IN BANACH SPACES AND SOME PROBLEMS IN NUMBER THEORY

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Abstract

We review some results in reordering of series in Banach spaces that have some applications in Analytic Number Theory.

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1 Introduction

The purpose of this note is to point out some problems in number theory, dealing mainly with properties of the Riemann zeta function ζ, for which the theory of reorderings of series in Banach spaces are (or could be) useful. First we review briefly some results of this theory and later we mention (probable) applications to number theory.

Our starting point is the following:

Theorem *Let* $\sum_{n\geq 1} t_n$ *be a convergent series of real numbers. Then precisely one of the following holds:*

(UC) For every bijection π *of* $\mathbb{N}$, $\sum_{n\geq 1} t_{\pi(n)}$ *converges and the resulting sum is* $\sum_{n\geq 1} t_n$. *Moreover, this occurs if and only if* $\sum_{n\geq 1} |t_n|$ *converges.*

(CC) There are bijections π *of* $\mathbb{N}$ *for which* $\sum_{n\geq 1} t_{\pi(n)}$ *is not convergent; in this case for each* $t \in \mathbb{R}$ *there is a bijection* π *of* $\mathbb{N}$ *such that* $t = \sum_{n\geq 1} t_{\pi(n)}$. *We can also find a bijection for which the resulting permuted sum diverges to* $+\infty$ *or* $-\infty$.

(UC) is due to G.Dirichlet (1837) and (CC) is due to B.Riemann (1854), with a small gap filled in by U.Dini (1868).

P.Levy (1905) proved that given a convergent series of complex numbers, the set of convergent rearrangements of the series is either a singleton, a line in $\mathbb{C}$, or the whole complex plane. He also gave an extension to $\mathbb{R}^k$, $k \geq 3$, that was not correct. E.Steinitz (1913) found a proper formulation and a correct proof of the Lévy-Steinitz theorem: For any convergent series in $\mathbb{R}^k$, the set of sums of its convergent rearrangements is an affine set, that is, the translate of a linear subspace.

V.M.Kadets (1986) proved that in any infinite dimensional Banach space there is a convergent series which has the property that the set of sums of its convergent rearrangements fails to be convex.

M.I.Kadets and K.Wózniakowski (1989), P.A.Kornilov (1988) and P.Enflo (unpublished) proved, independently, that any infinite dimensional Banach space contains a convergent series such that each of its convergent rearrangements has one of two values.

W.Banaszczyk (1990, 1993) has shown that a Fréchet space is nuclear if and only if the Lévy-Steinitz theorem holds in it.

For the results quoted above (and much more), see [5].

The following result on conditionally convergent series in a Hilbert space appears in [8]:

Theorem 1. *Suppose that a series* $\sum_{n\geq 1} u_n$ *of vectors in a real Hilbert space* H *satisfies the condition* $\sum_{n\geq 1} \|u_n\|^2 < \infty$, *and for any* $e \in H$ *with* $\|e\| = 1$ *the series* $\sum_{n\geq 1} \langle u_n, e\rangle$ *is conditionally convergent (with some rearrangement of the terms if necessary). Then for any* $s \in H$ *there is a bijection* π *of* $\mathbb{N}$ *such that* $\sum_{n\geq 1} u_{\pi(n)} = s$ *in the norm of* H.

If T is a compact operator in the separable Hilbert space H and $\varphi = \{\varphi_i\}_{i\in\mathbb{N}}$ is an orthonormal basis in H, we say that $\varphi \in \mathrm{Dom}(\mathrm{tr}T)$ if the series $\sum_{i\geq 1}\langle T\varphi_i, \varphi_i\rangle$ converges. If $\varphi \in \mathrm{Dom}(\mathrm{tr}T)$ we denote the complex number $\sum_{i\geq 1}\langle T\varphi_i, \varphi_i\rangle$ by $\mathrm{tr}_\varphi T$. This correspondence defines a function from $\mathrm{Dom}(\mathrm{tr}T)$ into $\mathbb{C}$. The range of this function will be denoted by $\mathrm{R}(\mathrm{tr}T)$. It is possible to have $\mathrm{Dom}(\mathrm{tr}T) = \emptyset$ and therefore also $\mathrm{R}(\mathrm{tr}T) = \emptyset$. The following theorem was proven by A.Ben-Artzi (1984) [7], and it bears some resemblance to the Lévy theorem.

Theorem *Let* $T \in \mathrm{B}(H)$ *be compact,* H *a separable Hilbert space such that* $\mathrm{Dom}(\mathrm{tr}T) \neq \emptyset$. *Then*

(i) $\mathrm{R}(\mathrm{tr}T)$ *is either a point, or a straight line in the complex plane, or the whole complex plane.*

(ii) There exists an orthonormal basis $\varphi = \{\varphi_i\}_{i\geq 1}$, such that $\mathrm{R}(\mathrm{tr}T)$ *coincides with the set of all convergent rearrangements of the series* $\sum_{i\geq 1}\langle T\varphi_i, \varphi_i\rangle$.

(iii) The case when $\mathrm{R}(\mathrm{tr}T)$ *is a straight line occurs if and only if T can be represented as a sum $T = \mu A + R$ where R is a trace class operator, $\mu \in \mathbb{C}\setminus\{0\}$, $A = A^*$ and neither A_+ nor A_- is of trace class.*

Finally, the author of this note proved in 1996 [2] the following result about reorderings of harmonic like divergent series:

Theorem 2. *If $f : \mathbb{N} \to \mathbb{C}$ is a function such that $\sum_{n\geq 1}\left|f(n) - \frac{1}{n}\right| < \infty$ and $\alpha \in [0,1]$ is irrational, then*

$$\sum_{n\geq 1}\left\{f\left(\left[\frac{n}{\alpha}\right]\right) + f\left(\left[\frac{n}{1-\alpha}\right]\right) - f(n)\right\} = -\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha),$$

where $[x]$ denotes the integer part of x.

2 Applications to Number Theory

We now mention some problems in number theory, some still unsolved, for which the above results are or could be useful.

If $\rho(x) = x - [x]$ is the fractionary part function, in 1955 A. Beurling [6] proved the following

Theorem *If*

$$M = \{f : f(x) = \sum_{k=1}^{N} a_k\rho\left(\frac{\theta_k}{x}\right),$$

$$\sum_{k=1}^{N} a_k\theta_k = 0,\ 0 < \theta_k \leq 1,\ a_k \in \mathbb{C},\ 1 \leq k \leq N,\ N \geq 2\},$$

then the Riemann Hypothesis (R.H.) holds if and only if $\overline{M} = L^2(0,1)$. *Moreover,* $\overline{M} = L^2(0,1)$ *if and only if* $1 \in \overline{M}$.

If μ is the Möbius function, E.Meissel (1854) [9] has proven that if $x \geq 1$ then $\sum_{n\geq 1} \mu(n)\left[\frac{x}{n}\right] = 1$, and H. von Mangoldt (1897) [9] has shown that $\sum_{n\geq 1} \frac{\mu(n)}{n} = 0$. Combining these last two equations we get that pointwise it holds that

$$\sum_{n\geq 1} \mu(n)\left\{\rho\left(\frac{\theta}{nx}\right) - \frac{1}{n}\rho\left(\frac{\theta}{x}\right)\right\} = -\mathcal{X}_{]0,\theta]}(x),\ \forall \theta, x \in]0,1]. \qquad (2.1)$$

The partial sums of this series are in M, but the series does not converge strongly in $L^2(0,1)$ (R.Heath-Brown, private communication); but to establish R.H. weak convergence would suffice.

For any bijection π of $\mathbb{N}$ it can be shown that

$$\sum_{n\geq 1} \mu(\pi(n))\left\{\rho\left(\frac{\theta}{\pi(n)x}\right) - \frac{1}{\pi(n)}\rho\left(\frac{\theta}{x}\right)\right\} =$$
$$\left[\frac{\theta}{x}\right]\sum_{n\geq 1} \frac{\mu(\pi(n))}{\pi(n)} - \mathcal{X}_{]0,\theta]}(x),\ \forall \theta, x \in]0,1].$$

Since the series $\sum_{n\geq 1} \frac{\mu(n)}{n}$ is conditionally convergent, the permuted series $\sum_{n\geq 1} \frac{\mu(\pi(n))}{\pi(n)}$ can take any real value or even be divergent. Therefore since the function $g_\theta(x) = \left[\frac{\theta}{x}\right]$ does not belong to $L^2(0,1)$, a natural question to ask is: Are there any bijections π of $\mathbb{N}$ such that $\sum_{n\geq 1} \frac{\mu(\pi(n))}{\pi(n)} = 0$ and the series

$$\sum_{n\geq 1} \mu(\pi(n))\left\{\rho\left(\frac{\theta}{\pi(n)x}\right) - \frac{1}{\pi(n)}\rho\left(\frac{\theta}{x}\right)\right\} = -\mathcal{X}_{]0,\theta]}(x)$$

converges weakly in $L^2(0,1)$?

S.M.Voronin (1973) [8] has used theorem 1 to prove the following differential independence result for the Riemann zeta function.

Theorem *If $F(\zeta(s),\zeta'(s),\ldots,\zeta^{(N-1)}(s))=0$ identically in $s\in\mathbb{C}$, where F is a continuous complex function, then F is identically zero.*

B.Bagchi (1982) [10] has generalized this result proving that ζ does not satisfy a differential-difference equation: If $h_1<h_2<\cdots<h_m$ are real constants and

$$\Phi(\zeta(s+h_1),\ldots,\zeta^{(n_1)}(s+h_1),\ldots,\zeta(s+h_m),\ldots,\zeta^{(n_m)}(s+h_m))=0$$

where Φ is a continuous complex function, then Φ is identically zero.

In [1],[4] the present author proved that if $[A_\rho f](\theta)=\int_0^1\rho\left(\dfrac{\theta}{x}\right)f(x)dx$ is considered as an operator on $L^2(0,1)$, then R.H. holds if and only if $\mathrm{Ker}A_\rho=\{0\}$. Among other things we proved that A_ρ is Hilbert-Schmidt but neither nuclear nor normal, and we determined its spectrum, its (generalized) eigenvectors and its modified Fredholm determinant. If $\{\lambda_n\}_{n\geq 1}$ is the sequence of non-zero eigenvalues of A_ρ, ordered in such a way that $|\lambda_n|\geq|\lambda_{n+1}|$, $\forall n\in\mathbb{N}$, where each eigenvalue is repeated a number of times equal to its multiplicity, then $|\lambda_n|\leq\dfrac{e}{n}$ $\forall n\in\mathbb{N}$, $\sum\limits_{n\geq 1}|\lambda_n|=\infty$ and $\lambda_n\notin\mathbb{R}$ for an infinite number of n's. Two natural questions that arise are:

1) Is the series $\sum\limits_{n\geq 1}\lambda_n$ conditionally convergent? and if so, what alternative of Lévy's theorem holds for it?

2) Which alternative of Ben-Artzi's theorem applies to A_ρ?

We have shown that $\mathbb{R}\subset \mathrm{R}(\mathrm{tr}A_\rho)$ which implies that either $\mathrm{R}(\mathrm{tr}A_\rho)=\mathbb{R}$ or $\mathrm{R}(\mathrm{tr}A_\rho)=\mathbb{C}$.

The Beurling function defined by

$$J(\alpha)=\int_0^1\rho\left(\frac{1}{x}\right)\rho\left(\frac{\alpha}{x}\right)dx,\ \alpha\in[0,1]$$

appears in the Beurling approach to the R.H. [3]. Theorem 2 has been useful to establish some of the properties of this function listed below. J is continuous, not differentiable in $[0,1] \cap \mathbb{Q}$, and obeys the functional equation

$$\begin{aligned} -\frac{\alpha \ln \alpha}{2} - \frac{(1-\alpha)\ln(1-\alpha)}{2} &= J(1) - J(\alpha) - J(1-\alpha) + \alpha \\ &+ (1-\alpha) J\left(\frac{\alpha}{1-\alpha}\right), \ \forall \alpha \in [0, 1/2]. \end{aligned}$$

For J we have several expressions as an infinite series:

$$J(\alpha) = K(\alpha), \quad \forall \alpha \in [0,1] \setminus \mathbb{Q},$$

$$J(\alpha) = K(\alpha) + \frac{p+q}{q}\left\{\ln \Gamma\left(1 - \frac{1}{p+q}\right) - \frac{\gamma}{p+q}\right\}$$

if $\alpha = \dfrac{p}{q} \in [0,1] \cap \mathbb{Q}$, $p, q \in \mathbb{N}$, $(p,q) = 1$, where Γ is the gamma function, γ is the Euler constant and

$$\begin{aligned} K(\alpha) &= \frac{\ln(1+\alpha)}{2} + \frac{\alpha}{2}\ln\left(\frac{1+\alpha}{\alpha}\right) - \alpha + \\ &- (1+\alpha)\sum_{m\geq 1}\left\{\ln\left(1 - \frac{\rho(m\alpha)}{m(1+\alpha)}\right) + \frac{\rho(m\alpha)}{m(1+\alpha)}\right\} \\ &- (1+\alpha)\sum_{m\geq 1}\left\{\ln\left(1 - \frac{\rho(\frac{m}{\alpha})\alpha}{m(1+\alpha)}\right) + \frac{\rho(\frac{m}{\alpha})\alpha}{m(1+\alpha)}\right\} \end{aligned}$$

A formula that holds for all α in $[0,1]$ is

$$\begin{aligned} J(\alpha) &= -\frac{\alpha \ln \alpha}{2} - \sum_{n\geq 1}\left\{\left(\sum_{k=1}^{[\frac{n}{\alpha}]}\frac{1}{k}\right) - \ln\left[\frac{n}{\alpha}\right] - \gamma - \frac{1}{2[\frac{n}{\alpha}]}\right\} + \\ &- \sum_{n\geq 1}\left\{\ln\left(1 - \frac{\rho(\frac{n}{\alpha})}{\frac{n}{\alpha}}\right) + \frac{\rho(\frac{n}{\alpha})}{\frac{n}{\alpha}}\right\} - \frac{1}{2}\sum_{n\geq 1}\frac{\rho(\frac{n}{\alpha})}{[\frac{n}{\alpha}]\frac{n}{\alpha}} \\ &+ \frac{\alpha}{2}\{\ln(2\pi) - \gamma - 1\}. \end{aligned}$$

The series $\sum_{n\geq 1} \frac{\rho(n\alpha) - \frac{1}{2}}{n}$ converges almost everywhere and it defines a lower semicontinuous function. If $\alpha \in \mathbb{Q}$ then one of the series $\sum_{n\geq 1} \frac{\rho(n\alpha) - \frac{1}{2}}{n}$ or $\sum_{n\geq 1} \frac{\rho(\frac{n}{\alpha}) - \frac{1}{2}}{n}$ is divergent. We conjecture that the series $\sum_{n\geq 1} \frac{\rho(n\alpha) - \frac{1}{2}}{n}$ is convergent if and only if $\alpha \notin \mathbb{Q}$. If this is true, then we have that

$$\begin{aligned} J(\alpha) &= -\alpha + \frac{1+\alpha}{2}\{\ln(2\pi) - \gamma\} + \frac{1-\alpha}{2}\ln\alpha + \\ &- \alpha \sum_{m\geq 1} \frac{\rho(\frac{m}{\alpha}) - \frac{1}{2}}{m} - \sum_{m\geq 1} \frac{\rho(m\alpha) - \frac{1}{2}}{m}, \quad \forall \alpha \in [0,1] \setminus \mathbb{Q}. \end{aligned}$$

These formulae for J and the functional equation (2.2) are compatible by virtue of theorem 2.

References

[1] ALCÁNTARA-BODE, J. (1993). *An Integral Equation Formulation of the Riemann Hypothesis.* J.Integral Equations and Operator theory. 17 151-168.

[2] ALCÁNTARA-BODE, J. (1996). *Reorderings of some Divergent Series.* Pro-Mathematica, vol. X, n°19-20 5-8.

[3] ALCÁNTARA-BODE, J. (2000). *Some Properties of the Beurling Function.* Pro-Mathematica, vol. XIV, n°27-28 1-11.

[4] ALCÁNTARA-BODE, J. (2001). *An Algorithm for the Evaluation of Certain Fredholm Determinants.* J.Integral Equations and Operator Theory, 39 153-158.

[5] DIESTEL, J., JARCHOW, H. AND TONGE, A. (1995). *Absolutely Summing Operators.* Cambridge University Press, Cambridge.

[6] DONOGHUE, W. F. (1969). *Distributions and Fourier Transforms.* Academic Press, New York.

[7] GOHBERG, I., GOLDBERG, S. AND KRUPNIK, N. (2000). *Traces and Determinants of Linear Operators.* Birkhäuser Verlag, Basel.

[8] KARATSUBA, A.A. AND VORONIN, S.M. (1992). *The Riemann Zeta-Function.* Walter de Gruyter, Berlin.

[9] NARKIEWICZ, W. (2000). *The Development of Prime Number Theory.* Springer, Berlin.

[10] TITCHMARSH, E. C. (1986). *The Theory of the Riemann Zeta-Function.* Second edition revised by D.R.Heath-Brown, Clarendon Press Oxford.

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