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# UNIQUENESS OF EQUILIBRIUM IN PRODUCTIVE ECONOMIES

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## *Abstract*

*In this paper we discuss the uniqueness of equilibria in productive economies, and we show that the source of the multiplicity of equilibria lies in the consumption set. This result is well know in the literature, our only object in this work is to show a resume of this theme for the discussion in our seminar.*


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# 1 Introduction

In this paper we will discuss the uniqueness of equilibrium in production economies, nevertheless we will show also, that some of the main characteristics of the productive economies arise from the convexity properties, particularly the Hotelling Lemma and the Shepard Lemma follows directly from the convex set separation theorems.

We will show that in productive economies the weak axiom of revealed preference is not only a sufficient property to uniqueness, but it is in some sense a necessary condition. This proposition tells us that the satisfaction of the WARP on the consumption side of the economy guarantee the uniqueness of equilibrium for any production set, nevertheless this fact should not be interpreted as asserting that if for any excess demand, WARP is not satisfied then multiplicity of equilibria follows. On the contrary the contribution of the production side to the index tends to make it positive and the more so the flatter of the boundary of the technology set is.

Recall that the se of functions with WARP is a set of zero measure is the space of continuous functions, see [2].

# 2 Production Economies

A production economy will be synthetically describe by two objects, the production set and the excess demand function.

**Definition 1.** *We will denote by* $Y \subset R^l$ **de production set.** *This set will be assumed to satisfy four basic properties:*

i) $Y$ is closed.

ii) $Y \cap R^l_+ = 0$.

iii) $Y$ convex.

iv) $-R^l_+ \subset Y$.

The item (i) is merely technical, but (ii), (iii), and (iv) have substance. Condition (i) says that $Y$ includes its boundary. Thus the limit of a sequence of inputs-outputs vector is also feasible. Condition (ii) says that inactivity is feasible and that it is not possible to produce a positive amount of some commodity without using any input, i.e. there is no free lunch. Condition (iii) says that if $y_1, y_2 \in Y$ then it is possible to operate simultaneously $y_1$, and $y_2$ at half level. Finally (iv) is the free disposal of commodities, this property holds if always, it is possible the absorption of any additional amounts of the inputs without any reduction in inputs.

We will say that $Y$ is infinitely divisible if $y \in Y$, then $\frac{1}{n}y \in Y$.

**Proposition 1.** *If $Y$ is closed and infinitely divisible, and it is not possible to produce a positive amount of some commodity without using any imput, then $Y$ is bounded from above.*

**Proof:** Suppose that for each $z \in R^l$ there exists some $y \in Y$ such that $y > z$ consider an increasing sequence $y_n \in Y$, such that $||y_n|| \to \infty$. As $\frac{y_n}{||y_n||} \in S$, there exists a convergent subsequence, suppose that $\bar{y}$ is the limit of this subsequence. As $Y$ is closed $\bar{y} \in Y$ but this is not possible because $y$ has only nonnegative coordinates, i.e., the vector $y$ does not use any inputs.

In some circumstances it is appropriate to introduce the irreversibility condition. This condition says that if $y \in Y$ then $-y \notin Y$. It follows that the null vector can not be an interior point of $Y$.

**Definition 2.** *A lipschitzian function $f : S \to R^l$ will be an excess demand function if satisfies:*

i) $f(S)$ is bounded bellow, that is $f(S) > -ke$ for some $k \in R$.

ii) $pf(p) = \beta(p)$ for all $p \in S$, and

iii) if $p_n \to p$, $p_n \in S$, and $p \notin S$, that is $p^j = 0$ for some $j$, then $||f(p_n)|| \to \infty$.

Where $\beta(p) = sup\ pY$ is the profit function. The excess demand function in general cannot be specified independently of the production set $Y$, because the demand and supply of a consumer depends on firms' technologies.

An useful property of convex function is given in the following proposition:

**Proposition 2.** *Let $f : X \to R$ be convex, there are equivalent:*

*i) $f$ is bounded from above on an open set $X$.*

*ii) $IntD(f) \neq$ and $f$ is locally lipschitziana on $IntD(f)$.*

Being $\beta$ a concave function the hypothesis that $f$ is lipschitzian, place no direct restriction on $\beta$. In contrast the hypothesis of $f$ be a $C^1$ function has very strong restrictions on the technological set. Recall that a convex function is differentiable iff and only if there exist one and only one subgradient (see [1]) and, in this case differentiability of $\beta$ implies that $\partial\beta(p) = y$. This is the Hotelling Lemma.

A pair $(p, y) \in S \times Y$ is an equilibrium if demand equals supply and the production vector $y$ maximizes profits. Formally:

**Definition 3.** *The pair $(p, y) \in S \times Y$ is an equilibrium if:*

*(i) $y = f(p)$, and*

*(ii) $py = \beta(p)$*

In this work, $f$ and $Y$ are aggregate concepts, an economy will be represented by $\mathcal{E} = \{Y, f\}$. A disaggregate representations would consists on three ingredients:

i) A collection of firms specified by there production sets,

ii) a collection of consumer specified by his preferences and endowments,

iii) a profit distribution rule, that determines how profits of every firm is distributed among the consumers.

By part (i), the equilibrium production vector $y$ is uniquely determined by $p$. Because $pf(p) = \beta(p)$ always holds, an equivalent definition is : $p \in S$, *is an equilibrium price vector if and only if* $f(p) \in Y$. Recall that for a productive economy the Walras law take the form:

$$p(f(p) - y(p)) = p\left\{\sum_{i=1}^{n}[f_i(p) - \sum_{j=1}^{m}\theta_{ij}y_j(p)]\right\} = p\sum_{i=0}^{n}0 = 0$$

where $f_i(p)$ is the excess demand function of the i-agent, $y_j(p)$ i the supply function of the $j$ producer and $\theta_{ij}$ represents consumer i's share of producer j's profit.

A case of salient interest is when $Y$ exhibits constant return to scale, that is $Y$ is a cone. Then the profit function $\beta$ is identically zero in its domain, and therefore: $pf(p) = 0$ for all $p \in S$ In turns an exchange economy is a particular case of a productive economy such that $Y = -R^l_+$.

## 2.1 Some Derived Construction

Given a production set $Y$ we will introduce in this section the closely related concept of distance function, projection function, and profit function.

Let us define, the distance function $\gamma : R^l \to R$ as:

$$\gamma(x) = \frac{1}{2} min_{y \in Y} ||x - y||^2.$$

Suppose that $Y$ is closed and convex, then there exist one and only one point $\pi(x) \in Y$ such that

$$\gamma(x) = \frac{1}{2} ||x - \pi(x)||^2$$

in this way $\pi R^n \to Y$ is a new function, which we shall call the projection function.

**Proposition 3.** *The projection function $\pi$ satisfies:*

$$||\pi(x) - \pi(y)|| \leq ||x - y||$$

*for any $x, y \in R^l$. This means that the projection function is lipschitzian.*

*The distance function $\gamma : R^l \to R$ is differentiable and $\partial\gamma(x) = x - \pi(x)$ at any $x \in R^l$.*

**Proof:** The properties of $\pi$ are general convexity facts. To see that $\gamma$ is a convex function: Let $x, z \in R^l$ and denote $x' = \pi(x), z' = \pi(z)$.

Then for all $0 \leq t \leq 1$ it follows that:

$$\gamma(tx + (1-t)z) \leq \frac{1}{2}\left(||(tx + (1-t)z) - (tx' - (1-t)z)||\right)^2$$

and now using the triangle inequality:

$$\gamma(tx + (1-t)z) \leq \left(t||x - x'|| + (1-t)||z - z'||\right)^2$$

holds. From the convexity of the square function and the definition of $\gamma$, the claim follows.

To see that $x - \pi(x)$ is the gradient of *gamma* suppose without loss of generality that: $\pi(x) = 0$ and let $H(x)$ be the supporting hyperplane at 0. Choose $v \in R^l$ and $\alpha \in R$ such that $x + v - \alpha x \in H(x)$, then $\alpha = \frac{(x+v)}{||x||^2}$. So,

$$\gamma(x+v) \geq ||\alpha x||^2 = \frac{1}{2}||x||^2 + vx + \frac{1}{2}\frac{(v \cdot x)^2}{||x||^2} \geq \gamma(x) + v \cdot [x - \pi(x)].$$

[·]

In the next proposition the main properties of the profit function are gathered:

**Proposition 4.** *The profit function $\beta : S \to R$ satisfies:*

*(i) The set $y(p) = \{y \in Y : py = \beta(p)\}$ (this is the supply correspondence) is nonempty and compact if $p \in S$.*

*(ii) $\beta$ is linear homogeneous, convex and continuous function. If $0 \in Y$ then $\beta$ is also nonnegative.*

*(iii) The gradient vector $\partial\beta(p)$ exists if and only if $\beta(p) = py$ for a unique $y \in$, and in this case $\partial\beta(p) = y$. (this is the Hotelling Lemma).*

**Proof:**

(i) If the set $Y$ is bounded from above, $0 \in Y$ and $p \in S$ then the closed subset $A = \{y \in Y : py > 0\}$ is bounded from below and hence a compact set. This implies that some production plan maximize the profit function, and then $y(p)$ is a non-emty set..

(ii) To prove the convexity of $\beta$ suppose that $y \in y(\alpha p + (1-\alpha)p')$. Then $\beta(\alpha p + (1-\alpha)p') = \alpha p y + (1-\alpha)p'y \leq \alpha\beta(p) + (1-\alpha)\beta(p')$. The continuity follows from the fact that: Any convex function $f$ on a finite dimensional space $X$ is continuous on $IntD(f)$ the interior of its domain.

iii) (This is a prove of the Hotelling's Lemma) Recall that the set of subgradients at $x$ $\partial f(x)$, of a continuous convex function is non empty. A functional $x^*$ is called a subgradient of $f$ at $x$ if: $f(y) - f(x) \geq x^*(y-x)$, $\forall y \in X$. And a convex function $f$ is differentiable at $x$ in the interior of its domain if and only if has a unique subgradient at $x$, in this case $\partial f(x) = f'(x)$.

Let us consider $y^*$ in the boundary of $Y$, $(\delta Y)$. Then there exist $p^*$ *such that* $p^*y^* = max_{y \in Y} p^*y$. To see this consider $P_Y = \{y \in R^l : y > y^*\}$. It is easy to see that $P_Y$ is a convex set such that $P_Y \cap Y = \emptyset$. Then we can invoke the separating hyperplane theorem to establish that there exists $p^*$ such that:

$$p^*y \leq c \leq p^*z \;\; \forall y \in Y, \;\; z \in P_Y,$$

Because $z$ can be chosen to be arbitrarily close to $y^*$ we conclude that: $p^*y \leq p^*y^* \leq c \leq p^*z$. Consequently, $\beta(p^*) = p^*y^*$.

As it is easy to see if $y^* \in y(p^*)$ then $y^*$ is a subgradient for $\beta$ at $p^*$. To see the reciprocal claim, let $q$ be a subgradient of $\beta$ such that $q \notin y(p^*)$, this means that $\beta(p^*) > p^*q$ i.e., $\beta(p^*) - p^*q > 0$. Choose $\lambda > 0$ such that $\lambda[\beta(p^*) - p^*q] - [\beta(p^*) - p^*q] < 0$. Consider $p = \lambda p^*$, thus from the linear homogeneity of $\beta$ it follows that $\beta(p) - \beta(p^*) < q(p - p^*)$, this contradicts our assumption that $q$ is a subgradient. So, $\beta$ is differentiable if and only if there there exists one and only one $y \in y(p)$.

To see the last part of our claim (iii), let us consider $\Phi(p) = \beta(p) - py^* \geq 0$ with equality if and only if $p = p^*$. If $\beta$ is a differentiable function, then $\Phi'(p^*) = \beta'(p^*) - y^* = 0$ and (iii) follows

**NOTA (1):** It is straightforward that if $Y$ is strictly convex, then there is an unique $y^*$ such that maximize $py$ s.t. $y \in Y$ and the the Hotelling's lemma holds.

**NOTA (2):** A type of production set of importance is the associate with a linear activity model. Observe that if $Y = Y_j \cap Y_{-j}$ where $Y_j$ linear, that is there exists $a \in R^l$ that

$$Y_j = \{y = \alpha a;\ \alpha > 0\} - R^l_+$$

Observe that in this case de function $\beta : S \to R$ is not a continuous continuous function in the interior of $Y_j$, and then is not differentiable, in this case the the Hotelling's lemma does not hold.

**Proposition 5.** *The relationships of $\beta$ with $\gamma$, $\pi$ are the following:*

*(i)* $\beta(x - \pi(x)) = (x - \pi(x))\pi(x)$ *for all* $x$.

*(ii)* *If* $py = \beta(p)$ *and* $y \in Y$ *then* $y = \pi(p + y)$

*(iii)* *If* $(p, y)$ *is an equilibrium pair, then* $py = \beta(p)$

**Proof:**

i) If $y \in \beta(p)$ then $y$ is in the boundary of $Y$. Suppose otherwise: then there is $y' \in Y$ such that $y' > y$; $py' > py$ contradicting the assumption that $y$ is profit maximizing. Let us now define $c = (x - \pi(x))\pi(x)$. Let $H = \{z \in R^l : (x - \pi(x))z = c\}$ be the supporting hyperplane, then $(x - \pi(x))w \leq c\ \forall w \in Y$, so $\beta(p) = (x - \pi(x))\pi(x)$.

ii) It follows that $y$ is in the boundary of $Y$ and $pz \leq py$, $\forall z \in Y$, then

$$H_p = \{z \in R^l : pz = py\}$$

is the supporting hyperplane on $y$. Now consider $x = p + y$ and item (i).

# 3 The Uniqueness of Equilibrium

In this section we show that if the excess demand function satisfies the weak axiom of revealed preference, then the equilibrium is unique. This conclusion is true also for interchange economies. But in the context of productive economies the weak axiom is not only a sufficient condition (with regularity) is also a indispensable one.

Nevertheless we will show that, the weak axiom is a necessary condition to obtain uniqueness for a given excess utility function for all regular technological set associate with this function, but for a given technological set, it is possible to obtain uniqueness without the weak axiom. Moreover the possibility of uniqueness is greater in production economies the more flatter is the boundary of this set.

Recall that the weak axiom is a rare property, because the subset of functions that satisfy this axiom is rare in the set of continuous functions, see [2]. This means that typically a excess demand function does not satisfies the WARP.

## 3.1 Uniqueness from WARP

A excess demand function $f$ satisfies the so called **weak axiom of revealed preference** if for all $p, p'$ in the domain of $f$ $f(p) \neq f(p')$ and $pf(p') \leq p'f(p')$ imply $p'f(p) > p'f(p)$.

**Proposition 6.** *Suppose that $\mathcal{E} = \{Y, f\}$ has a least one regular equilibrium, and that $f$ satisfy WARP, then there is a unique regular equilibrium*

Recall that an equilibrium $p$ is regular, if there exist a neighborhood of $p$ where there is not another equilibrium.

**Proof:** Let $p_1$ and $p_2$ be two equilibria. Take any $p_3 = \alpha p_1 + (1-\alpha)p_2$ $0 < \alpha < 1$. By convexity of $\beta$, $\beta(p_3) \leq \alpha\beta(p_1) + (1-\alpha)\beta(p_2)$. Because $\beta(p_3) = (\alpha p_1 + (1-\alpha)p_2)f(p_3)$ so either: $p_1 f(p_3) \leq \beta(p_1)$ or $p_1 f(p_3) \leq \beta(p_2)$. Say that $p_1 f(p_3) \leq \beta(p_1)$. Then:

(i) If $f(p_1) \neq f(p_3)$ by the weak axiom $p_3 f(p_1) > p_3 f(p_3)$ and then $f(p_1) \notin Y$, and then $p_1$ is not an equilibrium.

(ii) Hence, $f(p_1) = f(p_3)$ and then the equilibrium in not regular.

**Proposition 7.** *Let $f : S \to R^l$ be a $C^1$ excess demand function, with with $pf(p) = 0$ for all $p \in S$. If $f$ does not satisfy the WARP, then there is a constant returns production set $Y$ such that the economy $\mathcal{E} = \{Y, f\}$ has several equilibria.*

**Proof:** Let $f(p) \neq f(p')$, $pf(p') \leq pf(p) = 0$, $p'f(p) \leq p'f(p')$ Then we can take:

$$Y = \{y \in R^l : py \leq 0 \; and \; p'y \leq 0\}$$

Then the prices $p, p'$ are distinct equilibria for $\mathcal{E} = \{Y, f\}$.

Then WARP is the most general condition on the consumption side that guarantee the uniqueness of equilibrium at a regular economy for any production set. In particular as in general, gross substitutes (GS) does not imply WARP, this property is compatible with multiple regular equilibria. (Recall that, GS implies WARP when the number of goods, $n$, is less than four. There are counterexamples when $n \geq 4$. See [3])

## 3.2 Uniqueness from the Index Theorem

It is of interest to express the equilibrium condition $f(p) \in Y$ as the solution of a equation system:

The function $f(p) \in Y$ only if putting $z = p + f(p)$ we have $\pi(z) - f(z - \pi(z)) = 0$. Because in this case $\pi(p + f(p)) = f(p)$ and then $f(p) \in Y$.

Hence the function $G(z) = f(z - \pi(z))$ is well defined on an open subset of $R^l$ thus, the following proposition is straightforward:

**Proposition 8.** *Whenever the equation system:*

$$\pi(z) - G(z) = 0, \; \|z - \pi(z)\| = 1$$

*is satisfied, the price vector $p = z - \pi(z)$ is an equilibrium.*

**Proof:** Suppose that $\pi(z) = G(z)$ i.e., $\pi(z) = f(z - \pi(z))$. Consider $p = z - \pi(z)$ it follows that $\pi(z) = f(p)$ then $f(p) \in Y$ that is $p$ is an equilibrium price.

**Proposition 9.** *If $p$ is an equilibrium and $z = p + f(p)$, then the linear maps $\partial\pi(z)$ and $\partial G(z)$ take values in $T_p = \{v : pv = 0\}$.*

**Proof:**

(i) For all $v \notin Y$ we have that $\frac{1}{2}||v - G(v)||^2 - \gamma(v) \geq 0$. Let $v = z = p + f(p)$ then $\pi(z) = f(p)$, and then $z - \pi(z) = p$. From $G(z) = f(z - \pi(z))$ it follows that $G(z) = f(p) = \pi(z)$. This means that $\frac{1}{2}||z - G(z)|| - \gamma(z) = 0$ then the function $\frac{1}{2}||v - G(v)||^2 - \gamma(v)$ reaches the minimum at $v = z = p + f(p)$. Thus, taking derivative: $[z - G(z)][I - \partial G(z)] - \partial\gamma(z) = 0$ We know that $\partial\gamma(z) = [z - \pi(z)]$ and as $G(z) = \pi(z)$ it follows that $[z - \pi(z)]\partial G(z) = 0$ and $p\partial G(z) = 0$ holds. So, rank $\partial G(z) \subseteq T_p$.

(ii) Let $v$ be a vector in the direction $x - \pi(x)$, so $\pi(x + v) = \pi(x)$. Consequently, $\partial\pi(x)v = 0$ that is rank of $\pi(x) \subseteq T_p$.

From this proposition it follows that

$$\partial\pi(z)p = \partial G(z)p = 0$$

and so, rank of $(\partial\pi(z) - \partial G(z)) \leq l - 1$ [.]

The definition of regularity is the following:

**Definition 4.** *The equilibrium $p$ is regular if, putting $z = p + f(p)$, the linear map $(\partial\pi(z) - \partial G(z))$ has rank $l - 1$*

We will say that the production economy is regular if every equilibrium is regular.

**Proposition 10.** *Let $\bar{p}$ be an equilibrium price vector and $q \in R^l_+$, $q \neq 0$. Then $\bar{p}$ is regular if:*

$$\begin{vmatrix} \partial\pi(z) - \partial G(z) & q \\ -q^T & 0 \end{vmatrix} \neq 0$$

*Furthermore, the sign of the above determinant is independent of the particular $q$ chosen.*

Recall that the index of an equilibrium $p$ is given by:

$$index\ p = sign \begin{vmatrix} \partial\pi(z) - \partial G(z) & p \\ -p^T & 0 \end{vmatrix}$$

Let $p$ be a regular equilibrium price. We put $index\ p = +(-)1$ according to whether:

$$\begin{vmatrix} \partial\pi(z) - \partial G(z) & p \\ -p^T & 0 \end{vmatrix} > (<)0.$$

Now showing that there exists a function $\eta : R^l \to R^l$ such that:

(a) $\eta(z) = 0$ if and only if $\|z - \pi(z)\| = 1$ and $\pi(z) = G(z)$, this means that $p(z) = \|z - \pi(z)\|$ is an equilibrium, see proposition (8).

(b) If $\eta(z) = 0$, then $\partial\eta(z)$ exists, it is nonsingular, and letting $p(z) = \|z - \pi(z)\|$, $\|\partial\eta(z)\| = index\ p(z)$.

(c) For $r$ sufficiently large the function $\eta(z)$ restricted to $S_r^{l-1} = \{z : \|z\| = r\}$ points outward (this means that $\eta(z).g(z) > 0$ on the boundary of $Sr^{l-1}$, where $g(z)$ is the Gauss map.)

Then from the Poincare-Hopf theorem , we obtain the following proposition:

**Proposition 11.** *Let $E \subseteq S$ be the set of equilibrium price vector or the regular production economy $\mathcal{E} = \{Y, f\}$. Then $E$ is finite and $\sum_{p \in E} index\ p = 1$.*

This theorem asserts that there exists at least one equilibrium, and in all case the number of equilibria is odd.

The proof of this proposition is given in [4].

We know that the distance function is $C^1$ convex and satisfies $\partial\gamma(x) = x - \pi(x)$. If $\gamma$ is $C^2$ it follows that $\partial\pi(x) = I - \partial^2\gamma(x)$, and this matrix will be symmetric and positive semidefinite.

**Proposition 12.** *If $\gamma$ is $C^2$ at $x$ then $\partial^2\gamma(x)$ and $\partial\pi(x) = I - \partial^2\gamma(x)$, are positive semidefinite. Moreover $\partial\pi(x)v = 0$ and $\partial^2\gamma(x)v = v$ for $v = x - \pi(x)$*

**Proof:** From 9 and from the definition of $\gamma$ the equalities $[x - \pi(x)][\partial\pi(x)] = 0$ $\partial\pi(x) = I - \partial^2\gamma(x)$ hold. The positive semidefiniteness of $\partial^2\gamma(x)$ is a consequence of the convexity of $\gamma$. Using the Taylor's formula for $\gamma(x)$ until the second order approach it follows that $\partial^2\gamma(x)(v,v) \leq \|v\|^2$ So, the semidefiniteness of $I - \partial^2\gamma(x)$ follows.

For $v$ in the direction $x - \pi(x)$ it is obvious that $\pi(x+v) = \pi(x)$ then: $\partial\pi(x)v = 0$, and $\partial^2\gamma(x)v = v$ [.]

Because $G(x) = f(x - \pi(x))$, $f$ is lipschitzian, and $\partial^2\gamma(x) = I - \partial\pi(x)$, we have $\partial G(z)v = 0$ whenever $z = p + f(p)$; $\partial^2\gamma(z)v = 0$. Thus the map $L = \partial\pi(z) - \partial G(z)$, equals the identity on $M = \{v \in T_p : \partial^2\gamma(z) = 0\}$. $M$ represents the directions on which $Y$ is flat at $f(p)$. Therefore if $Y$ is completely flat at $f(p)$, that is $M = T_p$ then $L$ equals the identity on $T_p$.

Observe more flatter is $\delta Y$ at $y = f(p)$ closer is $\partial\pi(z) - \partial G(Z)$ to the identity map on $T_p$ in the limit case:

$$p = sign \begin{vmatrix} I & p \\ -p^T & 0 \end{vmatrix} = +1$$

Now applying the definition of regularity, there exists only one equilibrium price, irrespective of the form of $f$.

Where $\partial^2\beta(p)$ exists it follows that

$$. \, index\ p = sign \begin{vmatrix} \partial^2\beta(p) - \partial f(p) & p \\ -p^T & 0 \end{vmatrix}.$$

Because $\partial^2\beta(p)$ is positive semidefinte, if

$$p = sign \begin{vmatrix} -\partial f(p) & p \\ -p^T & 0 \end{vmatrix} > 0,$$

then

$$index\ p = sign \begin{vmatrix} \partial^2\beta(p) - \partial f(p) & p \\ -p^T & 0 \end{vmatrix} > 0.$$

Observe that if $min\partial^2\beta(p)(v,v)$ over $v \in T_p$ is large enough, (if the curvature od $\delta Y$ at $f(p)$ is small) then $\partial^2\beta(p) - \partial f(p)$ is positive quasi semidefinite on $T_p$ and so $indexp = 1$.

Then it follows that the source of multiplicity of equilibria lies in the consumption and no in the production side of the economy.

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