Pro Mathematica Vol. XVI, Nos. 31-32, 2002

HEADS REPRODUCTION IN HERCULES AND HYDRAS BATTLES

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Abstract

Hercules killed the Hydra of Lerna in a bloody battle—the second of the labor tasks imposed upon him in atonement for his hideous crimes. The Hydra was a horrible, aggressive mythological monster with many heads and poisonous blood, whose heads multiplied each time one of them was severed.

This paper explores some mathematical methods about this interesting epic battle. A generalization of the original Kirby & Paris model is proposed, concerning a general heads reproduction pattern. We also study the connection of this model with Goodstein ultra-growing and recursive sequences.

As an interesting application, we next analyze the inevitable death of another huge monster of our modern era: the Internet.

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1 The Legend of Hercules and the Hidra

Greek mythology has it that semi god Hercules[1], extramarital son of Jupiter[2] and the mortal Alcmene[3], killed his own wife and children during temporary insanity induced by goddess Juno[4], who was always determined to harm Hercules.

Upon awakening from temporary insanity, Hercules was appalled and repented from his deeds to god Apollo[5], asking him for forgiveness and spiritual guidance. The god of the Oracles forgave him and sent him to serve king Eurystheus[6] for twelve years, doing hard labor for his murders.

As part of the punishment imposed for these horrendous crimes, he was given twelve very difficult tasks that seemed impossible to fulfill. Fortunately, Hercules eventually had help from Hermes[7] and Athena[8] when he most needed it. Upon completion of these twelve tasks Hercules became, no doubt, the greatest of all Greek heroes.

The first task entrusted him was to kill the feared Lion of Nemea. The second task was perhaps the most dangerous of all: to kill the awful Hydra, a dangerous snake with venomous blood and many heads[9], which lived in a marsh of Lerna and terrorized the population. Hercules confronted the Hydra in a formidable battle where he cut off its heads with a sword. Each time Hercules severed a head, however, more heads sprang out in its place. According to official records, Hercules won.

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[1]Hercules (Latin name) or Heracles (Greek name), the most popular of Greek mythological heroes.

[2]Jupiter (Latin name) or Zeus (Greek name), father of gods. He conquered the Titans and defeated his father, Saturn. God of the heavens, daylight, time, and lightning.

[3]Beautiful mortal wife of Amphithryon.

[4]Juno (Latin name) or Hera (Greek name), is Jupiter's jealous wife and sister; daughter of Saturn; protectress of marriage.

[5]Apollo, god of the oracles, healing, poetry, the arts, the herds, the Day, and the Sun. Son of Jupiter and Leto.

[6]Eurystheus, king of Tiryns and Mycenae, was half-brother of Hercules.

[7]Hermes (Greek name) or Mercury (Latin name), son of Jupiter, god of cunning and theft, and messenger of the gods.

[8]Athena (Greek name) or Minerva (Latin name), daughter of Jupiter, goddess of wisdom, the arts, sciences, and industry.

[9]According to some mythological Greek biographers, the Hydra initially had eight mortal heads plus one immortal head. We didn't give credit here to these beliefs.

What the ancient Greeks did not reveal is that no matter in what order, or what strategy Hercules employed to cut off the heads, he would have always defeated the Hydra, although as we are about to see, he would have needed almost all the history of time to do it.

2 Mathematical Model

Although less charming, and botanically incorrect, we will mathematically represent the Hydra as a tree, and Hercules simply as an arrow pointing to one of the tree heads.

Definition 1. (Hydra, root, heads) *A Hydra H is a* tree*—that is, a finite graph, acyclic and connected, with a fixed node called* root *and denoted as* $\mathrm{root}(H)$*. Any terminal node differing from the root is called* head.

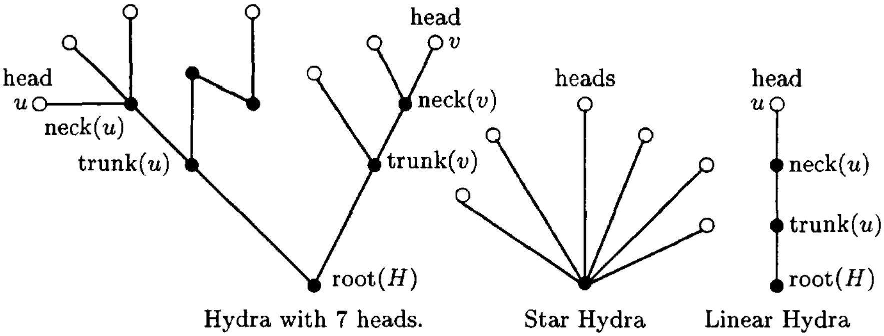


Figure 1: *Mathematical model and different types of Hydras. Star Hydras don't have trunks—only heads and root, which root coincides with the head's neck. Linear Hydras have only one head.*

The Hydra head reproductive process—after cutting one of them—can be defined in very diverse ways, some of which are more or less complex variations of the classic model we will explain next, which was introduced for the first time by Kirby & Paris [3] in 1982.

Definition 2. (Neck and trunk of a head) *Let v be a head of Hydra H. The predecessor node of v is called* neck *and is denoted by* $\mathrm{neck}(v)$*. If* neck(v) *has a preceding node on its way from the* root(H), *this node we call* trunk *of v and denoted it as* trunk(v). *In the event that* neck(v) = root(H), *we say that v has no trunk.*

The reproduction of Hydra's heads during the length of the battle is the most fascinating part of this monster. Associated to each head there's the body, which is the section of Hydra's anatomy that reproduces many times when the head is severed. Some heads have no associated body.

Definition 3. (Body of a head) *Let H be a Hydra and let v be a head of H. If v has a trunk, we define the* body *of v as the subtree containing the nodes* neck(v) *and all his successors, eliminating v and then adding the node* trunk(v) *as the root. In the event that v has no trunk, we say that v has no body.*

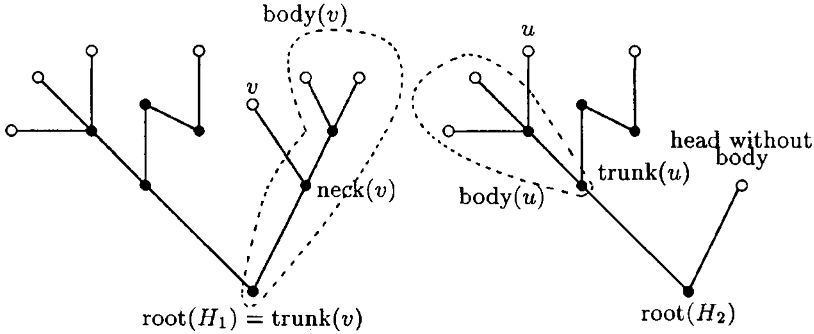


Figure 2: *Bodies of heads v and u of two different Hydras, H_1 y H_2.*

This is illustrated in Figure 2. Kirby & Paris classic head reproduction model is described in the following definition. An example of a stage of the battle is illustrated in Figure 3.

Definition 4. (Battle and head reproduction) *Let H be a Hydra. A* battle *between Hercules and Hydra H is a Hydra sequence $H_0 = H$, H_1, H_2, ... where Hydra H_n is obtained from the previous one, H_{n-1}, through the following reproductive scheme: Hercules cuts any given head $v \in H_{n-1}$; in response the Hydra adds n replicas of the body of v, from the* trunk(v) *on, if v has a body. In the event that the severed head has*

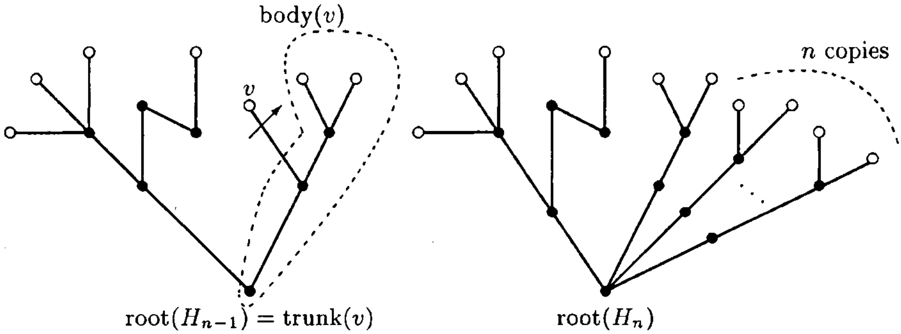


Figure 3: *Simulating stage n of a battle. At left, Hercules is about to cut the head v of Hydra H_{n-1}. At right we can see the new anatomy of Hydra H_n after reproducing n copies of the body of head v.*

no body, the Hydra won't reproduce at this stage. Hercules wins the battle if after finite number of stages k (the length of the battle), Hydra H_k is precisely the root.

Thus, the initial sequences of a typical battle between Hercules and the Hydra could develop, for example, as illustrated in Figure 4. In theory, Hercules has the possibility to choose which of the Hydra's heads to cut (in practice this would be a difficult task, with all the adrenaline spread out amidst the heat of battle!). In each stage, choosing a specific head's decapitation with respect to any other, establishes the various strategies availàble to Hercules.

As indicated above, Hydra not always produces new heads when one is cut. Reproduction takes place so long as the severed head has an associated body. But if the cut head lacks a body (as in star Hydras), no new heads are reproduced. This is illustrated in Figure 5.

Hydra's death will come after Hercules gets to cut off its last head, which won't reproduce since it lacks an associated body. We will say that one of Hercules strategies is a winner if the corresponding battle ends in the Hydra's death. In appearance, this situation will be hard to achieve due to the overwhelming growth in quantity of the Hydra's heads. That is why the following Kirby & Paris result [3] could seem disorienting at first.

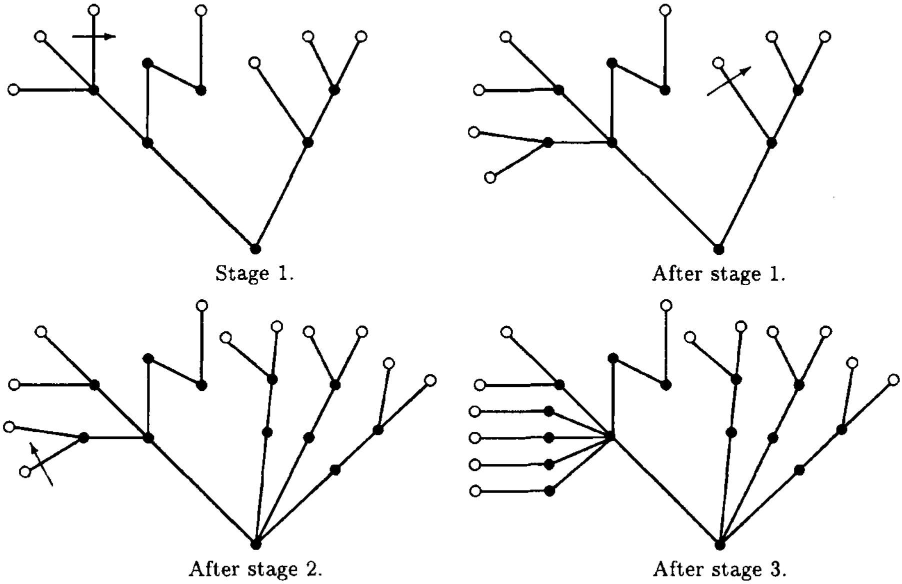


Figure 4: *First sequences of a typical battle between Hercules and the Hydra.*

Theorem 1. *Against any initial Hydra H_0, any Hercules strategy is a winning strategy.*

Amazing! The theorem tell us that in reality Hercules needs not employ intelligence to beat the Hydra, no matter how initially monstrous the Hydra might be. All he needs is patience and keeping himself alive in the heat of battle. Even more amazing is the following result, also owed to Kirby & Paris [3].

Theorem 2. *The affirmation, "Against any initial Hydra H_0, all Hercules recursive strategies are winning strategies", is an indemonstrable property in Peano's Arithmetic.*

That is, Theorem 1—even when true—cannot be demonstrated by using traditional arithmetic techniques. When talking about a *recursive strategy* we refer to a strategy whereby Hercules chooses heads according to some precise method or algorithm, which method—for example, does not allow random choices.

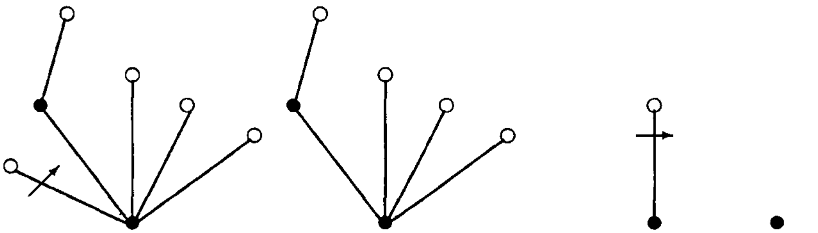


Figure 5: *At left, when cutting a bodiless head, the Hydra doesn't reproduce heads. At right,the Hydra's death occurs when the last head is severed.*

We will demonstrate later on (see Theorem 4) a generalization of Theorem 1. Theorem 2 demonstration is very complex, framing itself within the techniques of Model Theory and requiring the use of the concept "finite α-large sets", machinery developed by Ketonen & Solovay [2]. Kirby & Paris proved there exists at least one recursive strategy that, although being a winning one, cannot be demonstrated within Peano's Arithmetic. Simply said, we can affirm that the impossibility to demonstrate Theorem 1 within Peano's Arithmetic is due to the fact that there are recursive strategies where the number of necessary stages to kill the Hydra is immense—greater than the growth of primitive-recursive functions.

But let us first introduce a similar growing pattern, the interesting *Goodstein sequences*, used by Kirby & Paris in the proofs of these results.

3 Introduction to Goodstein Sequences

Let us start discussion with the following example. Number 266 allows only one binary representation:

$$266 = 2^8 + 2^3 + 2^1.$$

In the above representation, exponents 8 and 3 may also be represented in base 2, as well as the exponents of the exponents, etc. to obtain the *strict representation* in base 2 of 266:

$$266 = 2^{2^{2+1}} + 2^{2+1} + 2^1.$$

Then, from the principle of transfinite induction below ϵ_0, there exists $k \in \mathbb{N}$ such that $\alpha_k = 0$. That implies that $m_k = 0$. We need only to formalize some things:

1. *Primitive-recursiveness of Goodstein sequence* (m_k): Let

$$m = \sum_{i=0}^{s} n^i a_i = n^s a_s + n^{s-1} a_{s-1} + \cdots + n\, a_1 + a_0$$

be the normal representation of m in base n. We define the function $g^{m,n}\colon \mathbb{N} \mapsto \mathbb{N}$ by way of the recursion

$$g^{m,n}(x) := \begin{cases} \sum_{i=0}^{s} x^{g^{i,n}(x)} a_i & ,\text{if } m > 0 \\ 0 & ,\text{if } m = 0. \end{cases}$$

Thus, $g^{m,n}$ is primitive-recursive function and also $\{m\}_n = g^{m,n}(n)$. We define the Goodstein sequence (m_k) starting at base n by

$$\begin{aligned} m_0 &:= m, \\ m_1 &:= G_n(m_0) = g^{m_0,n}(n+1) - 1 \quad \text{and in general,} \\ m_{k+1} &:= G_{n+k}(m_k) = g^{m_k,n+k}(n+k+1) - 1, \quad \forall k \in \mathbb{N}. \end{aligned}$$

2. *Exact manner to associate ordinals to each* m_k: Consider the ordinal operator $\langle\alpha\rangle(n)$, defined for $\alpha \in \mathcal{O}\mathrm{rd}$, $n \in \mathbb{N}$, $\alpha < \epsilon_0$, inductively through

$$\begin{aligned} \langle 0\rangle(n) &:= 0, \\ \langle\beta+1\rangle(n) &:= \beta, \quad \text{and for } \delta > 0\text{:} \\ \langle\omega^\delta \cdot (\beta+1)\rangle(n) &:= \omega^\delta \cdot \beta + \omega^{\langle\delta\rangle(n)} \cdot n + \langle\omega^{\langle\delta\rangle(n)}\rangle(n). \end{aligned}$$

We can formally add ω to the domain of the functions $g^{m,n}$. Thus, we have:

$$\begin{aligned} m_0 &\longleftrightarrow \alpha_0 := g^{m_0,n}(\omega), \\ m_1 &\longleftrightarrow \alpha_1 := \langle\alpha_0\rangle(n), \quad \text{and in general,} \\ m_{k+1} &\longleftrightarrow \alpha_{k+1} := \langle\alpha_k\rangle(n+k), \quad \forall k \in \mathbb{N}. \end{aligned}$$

From definition, it is clear that (α_k) is a strictly decreasing sequence and also $\alpha_k < \epsilon_0$, for all $k \in \mathbb{N}$. Then, the principle of transfinite induction is applicable to (α_k), thereby concluding the proof of (a).

Idea of the proof of (b): Goodstein's theorem cannot be demonstrated in Peano's Arithmetic due—fundamentally— to the immense time the sequence (m_k) takes to reach 0. Indeed, let $h_n\colon \mathbb{N} \mapsto \mathbb{N}$ be the function defined by $h_n(m) := \mu k[m_k = 0]$, that is, the smallest index k for which Goodstein sequence (m_k) starting on base n reaches 0. Clearly, the functions h_n are recursive as well as total, for all $n \geq 2$, due to Goodstein's Theorem's veracity.

Ketonen & Solovay proved through indirect methods related to the *theory of indicators*, that under the supposition that Goodstein's Theorem be demonstrable in Peano's Arithmetic, then for any recursive and total function f there exists $n \in \mathbb{N}$ such that $f(x) < h_n(x)$, for x sufficiently large. This leads us to the absurd since we would come to construct a recursive and total function h growing faster than any recursive and total function. □

By the way, although the functions h_n are recursive and total, they are not primitive-recursive and therefore, they have a fast growth. For example, the standard sequence (starting on base 2) (4_k) reaches 0 to the astronomical index of $k = h_2(4) = 3 \times 2^{402\,653\,211} - 3$, a number of the order of $10^{121\,210\,700}$. The first terms of this sequence (4_k) are shown in Figure 6.

4 Ordinal Numbers Associated to Hydras

Definition 7. *Let H be a Hydra. We can associate H to an ordinal number $\alpha = \alpha(H)$ according to the following recursive procedure. To each node $s \in H$ we associate an ordinal number:*

(a) *If s is a head, we associate it to ordinal number 0.*

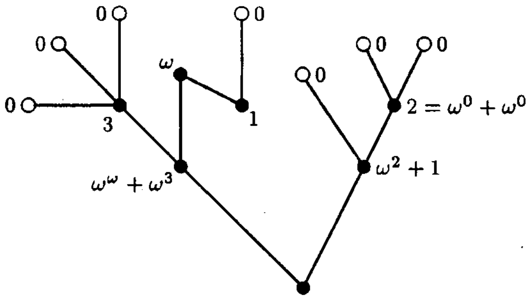
(b) *If s is not a head, then let's say that s has k immediate successor nodes. Let $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k$ be the ordinal numbers associated to these immediate successors. Then, we associate to s the ordinal number $\omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_k}$.*

Lastly, $\alpha(H)$ will be the associated ordinal to node $\mathrm{root}(H)$.

In Figure 7 we illustrate this procedure. Some comments must be made. First, observe that for any Hydra H we will have $\alpha(H) < \epsilon_0$ since H is a finite tree. Reciprocally, if $\alpha < \epsilon_0$ is any ordinal number, then

Goodstein sequence				*Ordinal α_k associated*
4_0	$=$	2^2	$\longleftrightarrow$	ω^ω
4_1	$=$	$3^3-1=3^2\cdot 2+3^1\cdot 2+2$	$\longleftrightarrow$	$\omega^2\cdot 2+\omega\cdot 2+2$
4_2	$=$	$4^2\cdot 2+4^1\cdot 2+1$	$\longleftrightarrow$	$\omega^2\cdot 2+\omega\cdot 2+1$
4_3	$=$	$5^2\cdot 2+5^1\cdot 2$	$\longleftrightarrow$	$\omega^2\cdot 2+\omega\cdot 2$
4_4	$=$	$6^2\cdot 2+6^1\cdot 2-1=6^2\cdot 2+6^1+5$	$\longleftrightarrow$	$\omega^2\cdot 2+\omega+5$
4_5	$=$	$7^2\cdot 2+7^1+4$	$\longleftrightarrow$	$\omega^2\cdot 2+\omega+4$
$\vdots$		$\vdots$		$\vdots$
4_9	$=$	$11^2\cdot 2+11^1+0$	$\longleftrightarrow$	$\omega^2\cdot 2+\omega$
4_{10}	$=$	$12^2\cdot 2+12^1-1$	$\longleftrightarrow$	$\omega^2\cdot 2+11$
4_{11}	$=$	$13^2\cdot 2+10$	$\longleftrightarrow$	$\omega^2\cdot 2+10$
$\vdots$		$\vdots$		$\vdots$
4_{21}	$=$	$23^2\cdot 2+0$	$\longleftrightarrow$	$\omega^2\cdot 2$
4_{22}	$=$	$24^2\cdot 2-1=24^2+24\cdot 23+23$	$\longleftrightarrow$	$\omega^2+\omega\cdot 23+23$
4_{23}	$=$	$25^2+25\cdot 23+25$	$\longleftrightarrow$	$\omega^2+\omega\cdot 23+22$
$\vdots$		$\vdots$		$\vdots$

Figure 6: *First terms of the Goodstein's standard sequence (4_k).*



$$\alpha(H)=\omega^{\omega^{\omega}+\omega^{3}}+\omega^{\omega^{2}+1}$$

Figure 7: *Procedure to associate an ordinal number to Hydra H.*

we can associate α to a Hydra $H = H(\alpha)$. Indeed, since $\alpha < \epsilon_0$, then Cantor's normal form is relevant to express α in an only way as

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_r},$$

with $\alpha > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$ and also, by expressing each ordinal $\alpha_1, \alpha_2, \ldots, \alpha_r$ by the same Cantor's normal form, the process is stabilized after a finite number of steps. For example, the following ordinal numbers α y β

$$\alpha = \omega^{\omega^{\omega+1}+\omega^3+1} + \omega^{\omega^2+1} + 1 \quad , \quad \beta = \omega^{\omega^\omega} + \omega^{\omega^3 \cdot 2+1} + \omega \cdot 3 + 1$$

define Hydras $H(\alpha)$ y $H(\beta)$ illustrated in Figure 8. It is clear that for any ordinal number $\alpha < \epsilon_0$, we will have the identities $\alpha(H(\alpha)) = \alpha$ and $H(\alpha(H)) = H$, if we identify as "equal" (or "equivalent") the Hydras differing only in the order of their nodes.

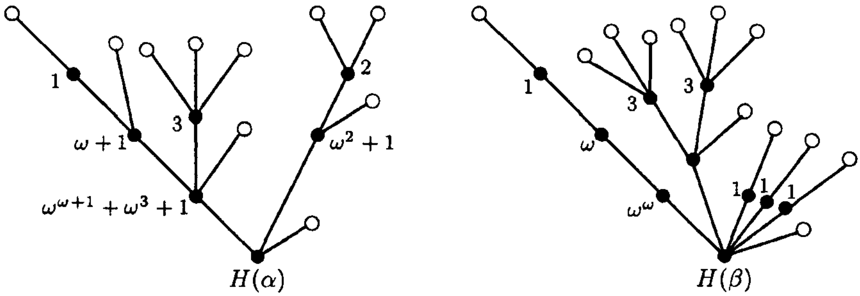
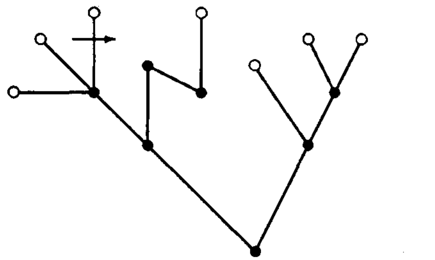


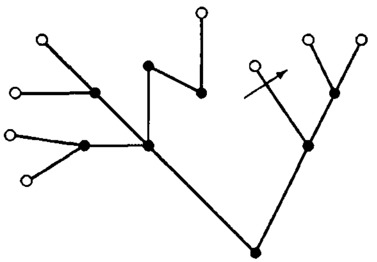
Figure 8: *Construction of Hydras corresponding to ordinal numbers α and β given.*

5 Slowest and Shortest Strategies

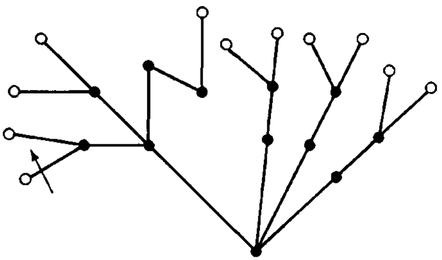
We have seen that in the battle between Hercules and the Hydra any strategy that Hercules employs is a winner. Which is the slowest strategy $\tau_{\max}$? And which is the shortest strategy $\tau_{\min}$? For a Hydra H, these "optimal" strategies are defined as follows:



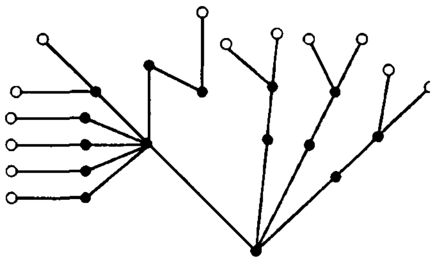
$\alpha_0 = \omega^{\omega^\omega + \omega^3} + \omega^{\omega^2+1}$



$\alpha_1 = \omega^{\omega^\omega + \omega^2 \cdot 2} + \omega^{\omega^2+1}$



$\alpha_2 = \omega^{\omega^\omega + \omega^2 \cdot 2} + \omega^{\omega^2} \cdot 3$



$\alpha_3 = \omega^{\omega^\omega + \omega^2 + \omega \cdot 4} + \omega^{\omega^2} \cdot 3$

Figure 9: *Sequence* (α_n) *of ordinals associated to Hydras in the battle in Figure 4.*

$\tau_{\max}$: Consists in choosing in each stage the head "closest" to the root; that is, to ascend from the root of H through the subtree associated to the smallest ordinal number. In case of draws, any head with the corresponding smallest ordinal number can be chosen.

$\tau_{\min}$: Consists in choosing in each stage the head "furthest" from the root; that is, to ascend from the root of H through the subtree associated to the greatest ordinal number. In case of draws, any head with the corresponding greatest ordinal number can be chosen.

Well, Loebl [4] demonstrated that while the strategy $\tau_{\min}$ produces battles of primitive-recursive duration (number of stages until the Hydra is defeated)—that is, 'short' battles, by contrast the strategy $\tau_{\max}$ produces battles of a duration superior to the growth of any primitive-recursive function—that is, "long" battles. On the other hand, Miserscque [7] demonstrated that to any Hydra H, the strategy $\tau_{\max}$ produces the longest battle between Hercules and the Hydra H. Precisely, the finitude of the battle when using this strategy $\tau_{\max}$ is indemonstrable in Peano's Arithmetic.

Matousek & Loebl [6] studied the following game of mixed strategies: Let's suppose the Hydra has a hidden helper, who tries to prolong battle as much as possible. Each time Hercules makes a move cutting a head (aiming to kill the monster as quickly as possible), he is followed by k different moves by the Hydra's hidden helper—the same kind of moves as Hercules but aiming to keep Hydra alive as long as possible. It was demonstrated that when $k = 1$ Hercules can keep the battle "short" (that is, of a primitive-recursive number of moves as a function of the number of nodes that the Hydra have), while if $k \geq 2$ the Hydra's hidden helper has a strategy to make battle "long" (duration superior to the growth of primitive-recursive functions, or finitude of the battle not demonstrable within Peano's arithmetic).

6 General Model for Heads Reproduction

In the classic model introduced by Kirby & Paris, the Hydra's heads spring out in arithmetic progression: in the first stage one copy comes out of the body of the cut head; in the second stage 2 copies are produces from the body, and 3 copies in the third stage, etc.

One of the first complaints the reader might voice is... How come such an organized head reproduction law—following arithmetic progression? Why not follow a geometric progression, or any other type of growth in this head reproduction matter?

The finitude of the battle between Hercules and the Hydra is not affected by replacing the classic model with an alternative one that allows for reproduction of an arbitrary number of heads. The following is the author's own result:

Theorem 4. *Suppose the law of head reproduction of the Kirby & Paris model is modified according to the following pattern: in the n-esimal stage after the cutting of a head, Hydra reproduces $f(n)$ replicas from the cut head's body, where $f\colon \mathbb{N} \mapsto \mathbb{N}$ is a total function. Then, for any initial Hydra H_0, all Hercules strategies are winners.*

Proof: The battle consists of a Hydra sequence $(H_n)_{n \in \mathbb{N}}$, to which we associate corresponding ordinal numbers $(\alpha_n)_{n \in \mathbb{N}}$, where $\alpha_n := \alpha(H_n)$, following the previously described procedure to assign ordinal numbers to Hydras.

Let τ be a Hercules strategy. We define the ordinal operator $[\alpha]_\tau(n)$ that calculates the ordinal number α_n in terms of the previous ordinal α_{n-1} and the natural number n (that is, $\alpha_n := [\alpha_{n-1}]_\tau(n)$) using the strategy τ. Let us demonstrate that (α_n) is a strictly decreasing sequence, thereby applying the principle of transfinite induction below ϵ_0 to deduce the existence of an index $k \in \mathbb{N}$ for which $\alpha_k = 0$, meaning that the corresponding Hydra H_k has died. In order to demonstrate that (α_n) is decreasing, it suffices to prove that for any $\alpha \in \mathcal{O}\mathrm{rd}$, with $0 < \alpha < \epsilon_0$ (that is, confronting any Hydra), and independently of the heads chosen by strategy τ, we will get

$$[\alpha]_\tau(n) < \alpha, \quad \forall n \in \mathbb{N}. \tag{6.1}$$

We will demonstrate this by using transfinite induction over α below ϵ_0, examining the five different forms that α could have according to ordinal number theory and the reproduction schema. First off, one should observe that for some relatively simple ordinals $\alpha < \epsilon_0$ the value of $[\alpha]_\tau(n)$ is independent of the chosen strategy τ. Indeed, out of the Hydra head reproduction definition, we will have the following particular cases:

$$\begin{aligned} [0]_\tau(n) &= 0, && (6.2)\\ [a+1]_\tau(n) &= a, && (6.3)\\ [\omega^{\gamma+1} \cdot (a+1)]_\tau(n) &= \omega^{\gamma+1} \cdot a + \omega^\gamma \cdot f(n), && (6.4) \end{aligned}$$

for all $\gamma < \epsilon_0$, $a \in \mathbb{N}$ and $n \in \mathbb{N}$. It is clear that the non-trivial particular cases (6.3), (6.4) comply with property (6.1). Next, suppose (as hypothesis of induction) that property (6.1) are true for all ordinals β smaller than α. Pending analysis are two possible forms that α can have. The first is when α has the form $\omega^\delta \cdot (a+1)$, where δ is a limit ordinal. In such case, we have:

$$[\omega^\delta \cdot (a+1)]_\tau(n) = \omega^\delta \cdot a + \omega^{[\delta]_\tau(n)}, \tag{6.5}$$

and being $\delta < \alpha$, the induction hypothesis is applicable, demonstrating property (6.1) for any strategy τ used. The last possibility occurs when the ordinal α has the form

$$\alpha = \omega^{\lambda_1} \cdot (a_1 + 1) + \cdots + \omega^{\lambda_i} \cdot (a_i + 1) + \cdots + \omega^{\lambda_r} \cdot (a_r + 1),$$

with $\alpha > \lambda_1 > \cdots > \lambda_r$ y $a_i \in \mathbb{N}$, $1 \leq i \leq n$. Strategy τ manifests itself in the choice of any of the subtrees associated to the previously added terms. Let's suppose that τ consists in choosing the subtree that starts with the i-esimal added term, $\omega^{\lambda_i} \cdot (a_i + 1)$. Then we have:

$$[\alpha]_\tau(n) = \omega^{\lambda_1} \cdot (a_1 + 1) + \cdots + \omega^{[\lambda_i]_\tau(n)} \cdot (a_i + 1) + \cdots + \omega^{\lambda_r} \cdot (a_r + 1). \quad (6.6)$$

But since $\lambda_i < \alpha$, we can apply the hypothesis of induction to obtain $[\lambda_i]_\tau(n) < \lambda_i$, from where property (6.1) is deduced. $\square$

The techniques used in this proof are similar to that used by Misercque in [7] and follows the central ideas of Kirby & Paris in [3]. It can be proved that in the generalized model just described, the number of moves it takes Hercules to kill the Hydra is recursive in f, as a function on the initial number of nodes.

Another interesting problem related to Kirby & Paris standard model about Hercules and the Hydra, is to examine whether there is a proof *within Peano's Arithmetic* for the finitude of recursive strategy $\tau_{\min}$. Well then, this is still an open problem!

In effect, Theorem 2 only establish that there *exists* a recursive strategy τ which does not admit finitude proof within Peano's Arithmetic. However, even in Kirby & Paris classic model the problem of proving the existence of recursive strategies whose finitude is demonstrable within usual arithmetic, could be as difficult as the famous conjecture $\mathcal{P} \neq \mathcal{NP}$, not yet resolved and fundamental in Computer Sciences.

Some particular cases of this problem have already been solved: Indeed, Luccio & Pagli in [5] studied the particular case of the alternative head reproduction scheme proposed in Theorem 4, when $f(n) = 2$ for all $n \in \mathbb{N}$ (that is, when the Hydra produces *two* copies of the cut head's body in each stage). They found combinatorial proof of $\tau_{\min}$ finitude within usual arithmetic by using the notion of *potential* of the Hydra's node. Their demonstration could extend to the case of any recursive, *bounded* and total function f.

7 Death of the Internet[11]

Let us now consider a different monster. Internet, a huge and intricate set of linked sites, engages for the bright future of global commerce. As a first approximation we regard the whole structure as a dynamic tree, where a user encounters more and more specialized pages moving from the root to the leaves, and such pages are updated continuously.

To discuss this phenomenon we shall pay a visit to a recent company at www.$o\lambda\iota\mu\pi o\sigma$.com, a paradigm of perfection in e.commerce and Internet evolution (Figure 10). Three sales departments have been established for some time under the supervision of Aphrodite[12] (Beauty Shop), Athena (Weapons), and Hermes (Hardware), and a new department has been opened recently under the supervision of Dionysus[13] (Wines & Liquors), accessible at the moment in just one leaf page of the $o\lambda\iota\mu\pi o\sigma$ tree.

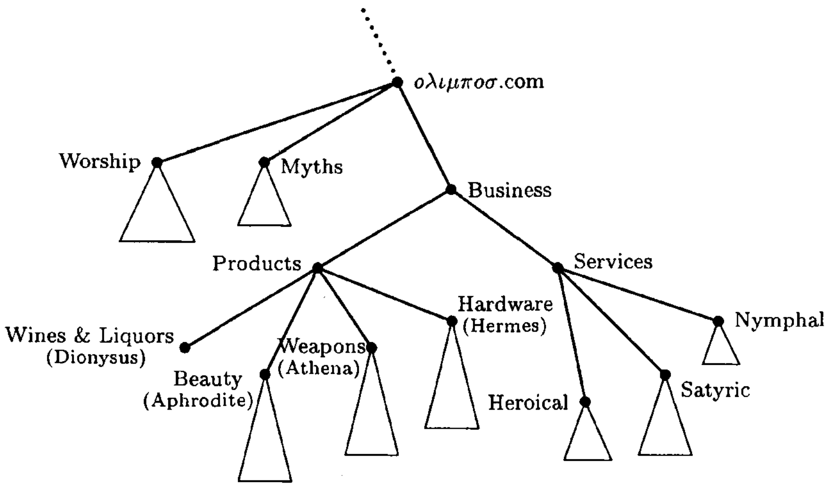


Figure 10: *The site* www.$o\lambda\iota\mu\pi o\sigma$.com

[11]Allegory by Fabricio Luccio and Linda Pagli [5] from Pisa University, Italy. Product of the disquisitions at a conference offered by the author on the theme of Hercules and the Hydra at Havana, Cuba, January, 1999.

[12]Aphrodite is the goddess of beauty and love, identified as Venus by the Romans.

[13]Dionysus is the Greek god of wine, son of Zeus and Sermele, identified as the Roman Bacchus.

Now spirits are in great demand, and the number of accesses to the Dionysus page has increased so rapidly that the company decides to upgrade the site. The Wines & Liquors page is moved one level up in the tree to be more directly accessible, thus becoming a child of Business and a sibling of Products. Some structuring is added to the department description, of the same type successfully experienced for Products. That is, a subtree like the one of Products is attached to Wines & Liquors.

So far so good. But the tree has undertaken a transformation of the Hydra type, in the version indicated in Theorem 4 when $f(n) = 1$. We then come to an inevitable conclusion: the Internet will disappear!

Epilogue. Yes, Internet will eventually disappear. However, as this conclusion may spread panic in Wall Street, we feel obligated to add a few words. First, the annihilation will take billions of steps, so there is no danger in the near future. Second, when a leaf directly attached to the root is cut off nothing grows from the tree, but a companion tree grows aside. A myriad of new Internets will then appear, then disappear, etc. Be prepared!

8 Acknowledgement

The author would like to thank Prof. Carlos Di Prisco of Universidad Central de Venezuela, for his valuable remarks and suggestions made to the manuscript of this paper.

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