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A STUDY ON TWO VARIABLE ANALOGUES OF CERTAIN FRACTIONAL INTEGRAL OPERATORS

Mumtaz ***Ahmad Khan*** *and*
Ghazi ***Salama Abukhammash***

Abstract

The Paper deals with a two variable analogues of certain fractional integral operators introduced by M. Saigo. Besides giving two variable analogues of earlier known fractional integral operators of one variable as special cases of newly defined operators, the paper establishes certain results in the form of theorems including integration by parts.

Department of Applied Mathematics, India

1 Introduction

The fractional calculus has been investigated by many mathematicians [14]. In their works the Reimann - Liouville operator (R-L) defined by

$$R^{\alpha}_{0,x}\, f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \tag{1.1}$$

was the most central, while Erdélyi and Kober defined their operator $(E-K)$ in connection with the Hankel transform [9] as

$$I^{\alpha,\eta}_{0,x}\, f = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, t^{\eta}\, f(t)\, dt. \tag{1.2}$$

Weyl and another Erdélyi-Kober fractional operators are defined as follows:

$$W^{\alpha}_{x,\infty}\, f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1}\, f(t)\, dt \tag{1.3}$$

and

$$K^{\eta,\alpha}_{x,\infty}\, f = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1}\, t^{\eta-\alpha}\, f(t)\, dt \tag{1.4}$$

respectively.

In 1978, M. Saigo [15] defined a certain integral operator involving the Gauss hypergeometric function as follows:

Let $\alpha > \beta$ and η be real numbers. The fractional integral operator $I^{\alpha,\beta,\eta}_{x}$, which acts on certain functions $f(x)$ on the interval $(0,\infty)$ is defined by

$$I^{\alpha,\beta,\eta}_{x}\, f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, F(\alpha+\beta, -\eta;\, \alpha;\, 1-\frac{t}{x})\, f(t)\, dt, \tag{1.5}$$

where Γ is the gamma function, F denotes the Gauss hypergeometric series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, |z| < | \tag{1.6}$$

and its analytic continuation into $|\arg(1-z)| < \pi$, and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Such an integral was first treated by Love [10] as an integral equation. However, if one regards the integral as an operator with a slight change, it will contain as special cases both $R-L$ and $E-K$ owing to reduction formulas for the Gauss function by restricting the parameters. The more interesting fact is that for this operator two kinds of product rules may be made up by virtue of Erdélyi's formulas [3], which were first proved by using the method of fractional integration by parts in the $R-L$ sense. From the rules, of course, the ones for $R-L$ and $E-K$ are deduced. Moreover, this operator is representable by products of $R-L's$ from which it is possible to obtain the integrability and estimations of Hardy-Littlewood type [4]. Saigo [15] also defined an integral operator on the interval (x,∞) as an extension of operators of Weyl and another Erdélyi - Kober operators as follows:

Under the same assumptions in defining (1.5), the integral operator $J_x^{\alpha,\beta,\eta}$ is defined by

$$J_x^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \cdot \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt. \tag{1.7}$$

Later on in 1988, Saigo and Riana [17] obtained the generalized fractional integrals and derivatives introduced by Saigo [15], [16] of the system $S_q^n(x)$ where the general system of polynomials

$$S_q^n(x) = \sum_{r=0}^{[n/q]} \frac{(-n)_{qr}}{r!} \ A_{n,r} \ x^r$$

were defined by Srivastava [18], where $q > 0$ and $n \geq 0$ are integers, and $A_{n,r}$ are arbitrary sequence of real or complex numbers.

2 Two Variable Analogues of Operators (1.5) and (1.7)

We define the two variable analogues of Saigo's operators (1.5) and (1.7) as follows:

I. Let $c > 0$, a, b, b' be real numbers. A two variable analogue of fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ due to M. Saigo is defined as

$$
\begin{aligned}
&{}_1I_{0,x;0,y}^{a,b,b';c} \quad f(x,y) \\
&= \frac{x^{-a}y^{-a}}{\{\Gamma(c)\}^2} \int_0^x \int_0^y (x-u)^{c-1}(y-v)^{c-1} F_1 \left[\begin{array}{c} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c; \end{array} \right] f(u,v)\,dvdu.
\end{aligned} \tag{2.1}
$$

SPECIAL CASES:

(i) For $a = b = b' = 0, c = \alpha$, (2.1) reduces to

$$
\begin{aligned}
&{}_1I_{0,x;0,y}^{0,0,0;\alpha} \; f(x,y) = {}_1R_{0,x;0,y}^{\alpha} \; f(x,y) \\
&= \frac{1}{\{\Gamma(\alpha)\}^2} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1} \; f(u,v) \; dvdu.
\end{aligned} \tag{2.2}
$$

Here (2.2) may be considered as a two variable analogue of Riemann - Liouville fractional integral operator $R_{0,x}^{\alpha}$.

(ii) For $a = c = \alpha, \; b = -\eta, \; b' = 0$, (2.1) becomes

$$
\begin{aligned}
&{}_1I_{0,\,x;\,0,\,y}^{\alpha,-\eta,0;\alpha} \; f(x,y) = {}_1^xE_{0,x;0,y}^{\alpha,\eta} \; f(x,y) \\
&= \frac{x^{-\alpha-\eta}y^{-\alpha}}{\{\Gamma(\alpha)\}^2} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1}u^{\eta} f(u,v)dvdu.
\end{aligned} \tag{2.3}
$$

(iii) For $a = c = \alpha, \; b = 0, \; b' = -\eta$, (2.1) gives

$$
\begin{aligned}
&{}_1I_{0,\,x;\,0,\,y}^{\alpha,0,-\eta;\alpha} \; f(x,y) = {}_1^y \; E_{0,x;0,y}^{\alpha,\eta} \; f(x,y) \\
&= \frac{x^{-\alpha}y^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^2} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1}v^{\eta} f(u,v)dvdu.
\end{aligned} \tag{2.4}
$$

Here (2.3) and (2.4) may be considered as two-variable analogues of Erdélyi-Kober fractional integral operator $E_{0,\,x}^{\alpha,\eta}$.

Under the same conditions of (2.1), a two variable analogue of another fractional integral operator $J_{x,\infty}^{\alpha,\beta,\eta}$ due to M. Saigo is defined as follows:

$$
\begin{aligned}
{}_1J_{x,\infty;y,\infty}^{a,b,b';c} f(x,y) &= \frac{1}{\{\Gamma(c)\}^2}\int_x^\infty\int_y^\infty (u-x)^{c-1}(v-y)^{c-1} \\
&\cdot F_1\left[\begin{matrix} a,b,b';1-\frac{x}{u},1-\frac{y}{v} \\ c; \end{matrix}\right] u^{-a}v^{-a}f(u,v)dvdu.
\end{aligned} \tag{2.5}
$$

Special Cases:

(i) For $a=b=b'=0,\ c=\alpha$, (2.5) reduces to

$$
\begin{aligned}
{}_1J_{x,\infty;y,\infty}^{0,0,0;\alpha} f(x,y) &= {}_1L_{x,\infty;y,\infty}^{\alpha} f(x,y) \\
&= \frac{1}{\{\Gamma(\alpha)\}^2}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}f(u,v)dvdu.
\end{aligned} \tag{2.6}
$$

It can be considered as a two variable analogue of Weyl fractional integral operator $L_{x,\infty}^{\alpha}$.

(ii) For $a=c=\alpha,\ b=-\eta,\ b'=0$, (2.5) becomes

$$
\begin{aligned}
{}_1J_{x,\infty;y,\infty}^{\alpha,-\eta,0;\alpha} f(x,y) &= {}_1^xK_{x,\infty,y,\infty}^{\alpha,\eta} f(x,y) \\
&= \frac{x^\eta}{\{\Gamma(\alpha)\}^2}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}u^{-\alpha-\eta}v^{-\alpha}f(u,v)dvdu.
\end{aligned} \tag{2.7}
$$

(iii) For $a=c=\alpha,\ b=0,\ b'=-\eta$ (2.5) gives

$$
\begin{aligned}
{}_1J_{x,\infty;y,\infty}^{\alpha,0,-\eta;\alpha} f(x,y) &= {}_1^yK_{x,\infty;y,\infty}^{\alpha,\eta} f(x,y) \\
&= \frac{y^\eta}{\{\Gamma(\alpha)\}^2}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}u^{-\alpha}v^{-\alpha-\eta}f(u,v)dvdu.
\end{aligned} \tag{2.8}
$$

Here (2.7) and (2.8) may be considered as two variable analogues of Erdélyi-Kober fractional integral operator $K_{x,\infty}^{\alpha,\eta}$.

II. Let $c>0,\ c'>0,\ a,b,b'$ be real numbers. Then a second two variable analogue of $I_{0,x}^{\alpha,\beta,\eta}$ is as given below:

$$
\begin{aligned}
{}_2I_{0,x;0,y}^{a,b,b';c,c'} f(x,y) &= \frac{x^{-a}y^{-a}}{\Gamma(c)\Gamma(c')}\int_0^x\int_0^y (x-u)^{c-1}(y-v)^{c'-1} \\
&F_2\left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c,c'; \end{matrix}\right] f(u,v)dvdu.
\end{aligned} \tag{2.9}
$$

SPECIAL CASES:

(i) For $a = b = b' = 0,\ c = \alpha,\ c' = \alpha'$, (2.9) reduces to

$$\begin{aligned} {}_2I^{0,0,0;\alpha,\alpha'}_{0,x;0,y} f(x,y) &= {}_2R^{\alpha,\alpha'}_{0,x;0,y} f(x,y) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\alpha')} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha'-1} f(u,v)\,dv\,du. \end{aligned} \tag{2.10}$$

Here (2.10) may be taken as second two variable analogues of Riemann-Liouville fractional integral operator $R^{\alpha}_{0,x}$. For $\alpha' = \alpha$, (2.10) reduces to (2.2).

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = 0,\ c' = \alpha'$, (2.9) becomes

$$\begin{aligned} {}_2I^{\alpha,-\eta,0;\alpha,\alpha'}_{0,x;0,y} f(x,y) &= {}^{x}_{2}E^{\alpha,\alpha',\eta}_{0,x;0,y} f(x,y) \\ &= \frac{x^{-\alpha-\eta}y^{-\alpha}}{\Gamma(\alpha)\Gamma(\alpha')} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha'-1} u^{\eta} f(u,v)\,dv\,du. \end{aligned} \tag{2.11}$$

For $\alpha' = \alpha$, (2.11) reduces to (2.3).

(iii) For $a = c = \alpha,\ b = 0,\ b' = -\eta,\ c' = \alpha$, (2.9) gives

$$\begin{aligned} {}_2I^{\alpha,0,-\eta;\alpha,\alpha'}_{0,x;0,y} f(x,y) &= {}^{y}_{2}E^{\alpha,\alpha',\eta}_{0,x;0,y} f(x,y) \\ &= \frac{x^{-\alpha}y^{-\alpha-\eta}}{\Gamma(\alpha)\Gamma(\alpha')} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha'-1} v^{\eta} f(u,v)\,dv\,du. \end{aligned} \tag{2.12}$$

For $\alpha' = \alpha$, (2.12) reduces to (2.4).

Here (2.11) and (2.12) may be taken as second two variable analogues of Erdélyi-Kober fractional integral operator $E^{\alpha,\eta}_{0,x}$.

Under the same conditions of (2.9), a second two variable analogue of $J^{\alpha,\beta,\eta}_{x,\infty}$ is as defined below:

$$\begin{aligned} {}_2J^{a,b,b';c,c'}_{x,\infty;y,\infty} f(x,y) &= \frac{1}{\Gamma(c)\Gamma(c')} \int_x^\infty \int_y^\infty (u-x)^{c-1}(v-y)^{c'-1} \\ & F_2\left[\begin{matrix} a,b,b';1-\frac{x}{u},1-\frac{y}{v} \\ c,c'; \end{matrix}\right] u^{-a}v^{-a} f(u,v)\,dv\,du. \end{aligned} \tag{2.13}$$

SPECIAL CASES:

(i) For $a = b = b' = 0,\ c = \alpha,\ c' = \alpha'$, (2.13) reduces to

$$
\begin{aligned}
{}_2J^{0,0,0;\alpha,\alpha'}_{x,\infty;y,\infty} f(x,y) &= {}_2L^{\alpha,\alpha'}_{x,\infty;y,\infty} f(x,y) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\alpha')}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha'-1} f(u,v)\ dvdu.
\end{aligned}
\tag{2.14}
$$

We may consider (2.14) as second two variable analogues of Weyl fractional integral operator $L^{\alpha}_{x,\infty}$. For $\alpha' = \alpha$, (2.14) reduces to (2.6).

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = 0, c' = \alpha'$, (2.13) becomes

$$
\begin{aligned}
{}_2J^{\alpha,-\eta,0;\alpha,\alpha'}_{x,\infty;y,\infty} f(x,y) &= {}^{x}_{2}K^{\alpha,\alpha',\eta}_{x,\infty;y,\infty} f(x,y) \\
&= \frac{x^{\eta}}{\Gamma(\alpha)\Gamma(\alpha')}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha'-1}u^{-\alpha-\eta}v^{-\alpha} f(u,v)\ dv\,du.
\end{aligned}
\tag{2.15}
$$

For $\alpha' = \alpha$, (2.15) reduces to (2.7).

(iii) For $a = c = \alpha,\ b = 0,\ b' = -\eta,\ c' = \alpha$, (2.13) gives

$$
\begin{aligned}
{}_2J^{\alpha,0,-\eta;\alpha,\alpha'}_{x,\infty;y,\infty} f(x,y) &= {}^{y}_{2}K^{\alpha,\alpha',\eta}_{x,\infty;y,\infty} f(x,y) \\
&= \frac{y^{\eta}}{\Gamma(\alpha)\Gamma(\alpha')}\int_x^\infty\int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha'-1}u^{-\alpha}v^{-\alpha-\eta} f(u,v)\ dvdu.
\end{aligned}
\tag{2.16}
$$

For $\alpha' = \alpha$, (2.16) reduces to (2.8).

We may consider (2.15) and (2.16) as second two variable analogues of Erdélyi-Kober fractional integral operator $K^{\eta,\alpha}_{x,\infty}$.

III. Let $c > 0, a,\ a',\ b,\ b'$ be real numbers. Then a third two variable analogue of $I^{\alpha,\beta,\eta}_{0,x}$ is as follows:

$$
\begin{aligned}
{}_3I^{a,a',b,b';c,}_{0,x;0,y} f(x,y) &= \frac{x^{-a}y^{-a}}{\{\Gamma(c)\}^2}\int_0^x\int_0^y (x-u)^{c-1}(y-v)^{c-1} \\
&F_3\left[\begin{matrix} a,a',b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c\ ; \end{matrix}\right] f(u,v)\,dvdu.
\end{aligned}
\tag{2.17}
$$

SPECIAL CASES:

(i) For $a = a' = 0,\ c = \alpha$, (2.17) reduces to

$$ {}_3I_{0,x;0,y}^{0,0,b;b',\alpha} f(x,y) = {}_1R_{0,x;0,y}^{\alpha} f(x,y) $$

$$ = \frac{1}{\{\Gamma(\alpha)\}^2} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1} f(u,v)\, dv\, du. $$

which is (2.2).

(ii) For $a = c = \alpha,\ a' = 0,\ b = -\eta$, (2.17) becomes

$$ \begin{aligned} {}_3I_{0,x;0,y}^{\alpha,0,-\eta,b';\alpha} f(x,y) &= {}_3^{x}E_{0,x;0,y}^{\alpha,\eta} f(x,y) \\ &= \frac{x^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^2} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1} u^{\eta} f(u,v)\, dv\, du. \end{aligned} \tag{2.18} $$

(iii) For $a = 0,\ a' = c = \alpha,\ b' = -\eta$, (2.17) gives

$$ \begin{aligned} {}_3I_{0,x;0,y}^{0,\alpha,b,-\eta;\alpha} f(x,y) &= {}_3^{y}E_{0,x;0,y}^{\alpha,\eta} f(x,y) \\ &= \frac{y^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^2} \int_0^x \int_0^y (x-u)^{\alpha-1}(y-v)^{\alpha-1} v^{\eta} f(u,v)\, dv\, du. \end{aligned} \tag{2.19} $$

Here (2.18) and (2.19) may be thought of as the third two variable analogues of Erdélyi-Kober fractional integral operator $E_{0,x}^{\alpha,\eta}$.

Under the same conditions of (2.17), a third two variable analogue of $J_{x,\infty}^{\alpha,\beta,\eta}$ is as defined below:

$$ \begin{aligned} {}_3J_{x,\infty;y,\infty}^{a,a',b,b';c} f(x,y) &= \frac{1}{\{\Gamma(c)\}^2} \int_x^\infty \int_y^\infty (u-x)^{c-1}(v-y)^{c-1} \\ & F_3 \left[\begin{matrix} a,a',b,b';1-\frac{x}{u},1-\frac{y}{v} \\ c'; \end{matrix} \right] u^{-a} v^{-a'} f(u,v)\, dv\, du. \end{aligned} \tag{2.20} $$

SPECIAL CASES:

(i) For $a = a' = 0$, and $c = \alpha$, (2.20) reduces to

$$ {}_3J_{x,\infty;y,\infty}^{0,0,b,b';\alpha} f(x,y) = {}_1L_{x,\infty;y,\infty}^{\alpha} f(x,y) $$

$$ = \frac{1}{\{\Gamma(\alpha)\}^2} \int_x^\infty \int_y^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1} f(u,v)\, dv\, du $$

which is (2.6).

(ii) For $a' = 0,\ a = c = \alpha,\ b = -\eta$, (2.20) becomes

$$
\begin{aligned}
{}_3J^{\alpha,0,-\eta,b';\alpha}_{x,\infty;y,\infty}\ f(x,y) &= {}^{x}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty}\ f(x,y) \\
&= \frac{x^{\eta}}{\{\Gamma(\alpha)\}^2}\int_x^{\infty}\int_y^{\infty}(u-x)^{\alpha-1}(v-y)^{\alpha-1}u^{-\alpha-\eta}\, f(u,v)\ dv\, du.
\end{aligned} \tag{2.21}
$$

(iii) For $a = 0,\ a' = c = \alpha,\ b' = -\eta$, (2.20) gives

$$
\begin{aligned}
{}_3J^{0,\alpha,b,-\eta;\alpha}_{x,\infty;y,\infty}\ f(x,y) &= {}^{y}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty}\ f(x,y) \\
&= \frac{y^{\eta}}{\{\Gamma(\alpha)\}^2}\int_x^{\infty}\int_y^{\infty}(u-x)^{\alpha-1}(v-y)^{\alpha-1}v^{-\alpha-\eta}\, f(u,v)\ dv\ du.
\end{aligned} \tag{2.22}
$$

Here (2.21) and (2.22) may be taken as the third two variable analogues of Erdélyi-Kober fractional integral operator $K^{\eta,\alpha}_{x,\infty}$.

IV Let $c > 0,\ c' > 0,\ a,\ b$ be real numbers. Then a fourth two variable analogue of $I^{\alpha,\beta,\eta}_{0,x}$ is as defined below:

$$
\begin{aligned}
{}_4I^{a,b;c,c'}_{0,x;0,y}\ f(x,y) &= \frac{x^{-a}y^{-a}}{\Gamma(c)\Gamma(c')}\int_0^x\int_0^y (x-u)^{c-1}(y-v)^{c'-1} \\
& F_4\left[\begin{matrix} a,b;1-\frac{u}{x},1-\frac{v}{y} \\ c,c';. \end{matrix}\right] f(u,v)\ dv\, du.
\end{aligned} \tag{2.23}
$$

Under the same conditions of (2.23), a fourth two variable analogue of $J^{\alpha,\beta,\eta}_{x,\infty}$ is as given below:

$$
\begin{aligned}
{}_4J^{a,b;c,c'}_{x,\infty;y,\infty}\ f(x,y) &= \frac{1}{\Gamma(c)\Gamma(c')}\int_x^{\infty}\int_y^{\infty}(u-x)^{c-1}(v-y)^{c'-1} \\
& F_4\left[\begin{matrix} a,b;1-\frac{x}{u},1-\frac{y}{v} \\ c,c'; \end{matrix}\right] u^{-a}v^{-a}\ f(u,v)\ dv\, du.
\end{aligned} \tag{2.24}
$$

3 In this Section Certain Theorems Involving the above Operators will be given

Theorem 3.1. *For functions* $f(x,y)$, $g(x,y)$, $f\left(\frac{1}{x},\frac{1}{y}\right)$ *and* $g\left(\frac{1}{x},\frac{1}{y}\right)$ *defined for* $0 \le x < \infty$, $0 \le y < \infty$ *and* $c > 0$, *we have*

$$\int_0^\infty \int_0^\infty (xy)^{a-c-1} f\left(\frac{1}{x},\frac{1}{y}\right) {}_1I_{0,x,0;y}^{a,b,b';c}\, g(x,y)dy\,dx$$
$$= \int_0^\infty \int_0^\infty (xy)^{a-c-1} g\left(\frac{1}{x},\frac{1}{y}\right) {}_1I_{0,x,0;y}^{a,b,b';c}\, f(x,y)\,dy\,dx \tag{3.1}$$

provided that each double integral exists.

Proof of Theorem 3.1 We have

$$\int_0^\infty \int_0^\infty (xy)^{a-c-1} f\left(\frac{1}{x},\frac{1}{y}\right) {}_1I_{0,x;0,y}^{a,b,b',c}\, g(x,y)\,dy\,dx$$

$$= \int_{x=0}^\infty \int_{y=0}^\infty (xy)^{a-c-1} f\left(\frac{1}{x},\frac{1}{y}\right) \frac{(xy)^{-a}}{\{\Gamma(c)\}^2}$$

$$\int_{u=0}^{u=x} \int_{v=0}^{v=y} (x-u)^{c-1}(y-v)^{c-1} F_1\left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c; \end{matrix}\right] g(u,v)\,dvdu\,dydx$$

$$= \int_{u=0}^\infty \int_{v=0}^\infty \int_{x=u}^\infty \int_{y=v}^\infty \frac{(xy)^{-c-1}}{\{\Gamma(c)\}^2} f\left(\frac{1}{x},\frac{1}{y}\right) (x-u)^{c-1}(y-v)^{c-1}$$

$$F_1\left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c; \end{matrix}\right] g(u,v)\,dydx\,dvdu$$

$$= \int_{p=0}^\infty \int_{q=0}^\infty \int_{t=p}^\infty \int_{w=q}^\infty \frac{(tw)^{-c-1}}{\{\Gamma(c)\}^2} f\left(\frac{1}{t},\frac{1}{w}\right) (t-p)^{c-1}(w-q)^{c-1}$$

$$F_1\left[\begin{matrix} a,b,b';1-\frac{p}{t},1-\frac{q}{w} \\ c; \end{matrix}\right] g(p,q)\,dw\,dt\,dq\,dp$$

(using the property of definite integrals that $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$).

We now make the substitution $p = \frac{1}{x}$, $q = \frac{1}{y}$, $t = \frac{1}{u}$ and $w = \frac{1}{v}$. Then the above becomes

$$= \int_{x=0}^{\infty} \int_{y=0}^{\infty} (xy)^{a-c-1} g\left(\frac{1}{x}, \frac{1}{y}\right) \frac{(xy)^{-a}}{\{\Gamma(c)\}^2}$$

$$\int_{u=0}^{u=x} \int_{v=0}^{v=y} (x-u)^{c-1} (y-v)^{c-1} F_1 \left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c; \end{matrix} \right] f(u,v)\; dvdu\; dydx$$

$$= \int_{x=0}^{\infty} \int_{y=0}^{\infty} (xy)^{a-c-1} f\left(\frac{1}{x}, \frac{1}{y}\right) {}_1I_{0,x;0,y}^{a,b,b',c} f(x,y)\; dy\, dx$$

This completes the proof of Theorem 3.1. Similarly, we can prove the theorems 3.2 to 3.8.

Theorem 3.2. *Under the conditions stated in theorem 3.1, we have*

$$\begin{aligned} &\int_0^{\infty} \int_0^{\infty} (xy)^{a-c-1} f\left(\frac{1}{x}, \frac{1}{y}\right) {}_1J_{x,\infty;y,\infty}^{a,b,b';c}\; g(x,y)\; dy\, dx \\ &= \int_0^{\infty} \int_0^{\infty} (xy)^{a-c-1} g\left(\frac{1}{x}, \frac{1}{y}\right) {}_1J_{x,\infty;y,\infty}^{a,b,b';c}\; f(x,y)\; dy\, dx \end{aligned} \tag{3.2}$$

provided that each double integral exists.

Theorem 3.3. *For functions* $f(x,y), g(x,y), f\left(\frac{1}{x}, \frac{1}{y}\right)$ *and* $g\left(\frac{1}{x}, \frac{1}{y}\right)$ *defined for* $0 \le x < \infty$, $0 \le y < \infty$ *and* $c > 0$, $c' > 0$, *we have*

$$\begin{aligned} &\int_0^{\infty} \int_0^{\infty} x^{a-c-1} y^{a-c'-1} f\left(\frac{1}{x}, \frac{1}{y}\right) {}_2I_{0,x;0,y}^{a,b,b';c,c'} g(x,y)\; dy\, dx \\ &= \int_0^{\infty} \int_0^{\infty} x^{a-c-1} y^{a-c'-1} g\left(\frac{1}{x}, \frac{1}{y}\right) {}_2I_{0,x;0,y}^{a,b,b';c,c'} f(x,y)\; dy\, dx \end{aligned} \tag{3.3}$$

provided that each double integral exists.

Theorem 3.4. *Under the conditions stated in theorem 3.3, we have*

$$\int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} f\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_2J^{a,b,b';c,c'}_{x,\infty;y,\infty}\, g(x,y)\; dy\, dx$$
$$= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} g\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_2J^{a,b,b';c,c'}_{x,\infty;y,\infty}\, f(x,y)\; dy\, dx \tag{3.4}$$

provided that each double integral exists.

Theorem 3.5. *For functions* $f(x,y)$, $g(x,y)$, $f\left(\frac{1}{x}, \frac{1}{y}\right)$ *and* $g\left(\frac{1}{x}, \frac{1}{y}\right)$ *defined for* $0 \le x < \infty, 0 \le y < \infty$ *and* $c > 0$, *we have*

$$\int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} f\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_3I^{a,a',b,b';c}_{0,x;0,y}\, g(x,y)\; dy\, dx$$
$$= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} g\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_3I^{a,a',b,b';c}_{0,x;0,y}\, f(x,y)\; dy\, dx \tag{3.5}$$

provided that each double integral exists.

Theorem 3.6. *Under the conditions stated in theorem 3.5, we have*

$$\int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} f\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_3J^{a,a',b,b';c}_{x,\infty;y,\infty}\, g(x,y)\; dy\, dx$$
$$= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1}\ g\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_3J^{a,a',b,b';c}_{x,\infty;y,\infty}\, f(x,y)\; dy\, dx \tag{3.6}$$

provided that each double integral exists.

Theorem 3.7. *For functions* $f(x,y)$, $g(x,y)$, $f\left(\frac{1}{x}, \frac{1}{y}\right)$ *and* $g\left(\frac{1}{x}, \frac{1}{y}\right)$ *defined for* $0 \le x < \infty$, $0 \le y < \infty$ *and* $c > 0$, $c' > 0$, *we have*

$$\int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} f\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_4I_{0,x;0,y}^{a,b,c,c'} \ g(x,y) \ dy\, dx$$
$$= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} \ g\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_4I_{0,x;0,y}^{a,b:c;c'} \ f(x,y) \ dy\, dx \tag{3.7}$$

provided that each double integral exists.

Theorem 3.8. *Under the conditions stated in theorem 3.7, we have*

$$\int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} f\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_4J_{x,\infty;y,\infty}^{a,b;c,c'} \ g(x,y) \ dy\, dx$$
$$= \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} g\left(\frac{1}{x}, \frac{1}{y}\right) \ {}_4J_{x\infty;y,\infty}^{a,b;c,c'} \ f(x,y) \ dy\, dx \tag{3.8}$$

provided that each double integral exists.

From these theorems certain interesting corollaries follow reedily for the operators (2.2), (2.3), (2.4), (2.6), (2.7), (2.8), (2.10), (2,11), (2,14), (2.15), (2.16), (2.18), (2.19), (2.21) and (2.22). Further, we can prove the following theorems:

Theorem 3.9. *If*

$$f(x,y) = \int_0^\infty \int_0^\infty (sx)^{\lambda-1} (ty)^{\mu-1} F_{\ell:m;n}^{p:q;r} \left[\begin{matrix} (a_p):(b_q);(c_r); \\ \\ (\alpha_\ell):(\beta_m);(\gamma_n); \end{matrix} \ sx, ty \right] g(s,t) dt\, ds$$

and $\Psi(x,y) = {}_1I_{0,x;0,y}^{a,b,b';c} \ f(x,y)$ *for* $c > 0$, *then we have*

$$\Psi(x,y) = \int_0^\infty \int_0^\infty s^{\lambda-1} t^{\mu-1} g(s,t)$$
$${}_1I_{0,x;0,y}^{a,b,b';c} x^{\lambda-1} y^{\mu-1} F_{\ell:m;n}^{p:q;r} \left[\begin{matrix} (a_p):(b_q);(c_r); \\ \\ (\alpha_\ell):(\beta_m);(\gamma_n); \end{matrix} \ sx, ty \right] dt\, ds \tag{3.9}$$

provided that the double integrals involved exist.

Proof of Theorem 3.9 We have

$$\Psi(x,y) = {}_1I^{a,b,b';c}_{0,x;0,y} f(x,y)$$

$$= \frac{(xy)^{-a}}{\{\Gamma(c)\}^2} \int_0^x \int_0^y (x-u)^{c-1}(y-v)^{c-1} F_1 \left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c\,; \end{matrix} \right] f(u,v)dv\,du$$

$$= \frac{(xy)^{-a}}{\{\Gamma(c)\}^2} \int_0^x \int_0^y (x-u)^{c-1}(y-v)^{c-1} F_1 \left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c\,; \end{matrix} \right]$$

$$\int_0^\infty \int_0^\infty (su)^{\lambda-1}(tv)^{\mu-1} F^{p:q;r}_{\ell:m;n} \left[\begin{matrix} (a_p):(b_q);(c_r); \\ & su,tv \\ (\alpha_\ell):(\beta_m);(\gamma_n); \end{matrix} \right] g(s,t)dt\,ds\,dv\,du$$

$$= \frac{(xy)^{-a}}{\{\Gamma(c)\}^2} \int_0^\infty \int_0^\infty s^{\lambda-1}t^{\mu-1}\ g(s,t)$$

$$\int_0^x \int_0^y u^{\lambda-1}v^{u-1}(x-u)^{c-1}(y-v)^{c-1}$$

$$F_1 \left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c\,; \end{matrix} \right] F^{p:q;r}_{\ell:m;n} \left[\begin{matrix} (a_p):(b_q);(c_r); \\ & su,tv \\ (\alpha_\ell):(\beta_m);(\gamma_n); \end{matrix} \right] dv\,du\,dt\,ds$$

$$= \int_0^\infty \int_0^\infty s^{\lambda-1}\ t^{u-1} g(s,t)$$

$${}_1I^{a,b,b';c}_{0,x;0,y}\ x^{\lambda-1}y^{\mu-1} F^{p:q;r}_{\ell:m;n} \left[\begin{matrix} (a_p):(b_q);(c_r); \\ & sx,ty \\ (\alpha_\ell):(\beta_m);(\gamma_n); \end{matrix} \right] dt\,ds.$$

This completes the proof Theorem 3.9. Similarly, we can prove theorem 3.10.

Theorem 3.10. *If*

$$f(x,y) = \int_0^\infty \int_0^\infty (sx)^{\lambda-1}(ty)^{\mu-1} F^{p:q;r}_{\ell:m;n} \left[\begin{matrix} (a_p):(b_q);(c_r); \\ & sx,ty \\ (\alpha_\ell):(\beta_m);(\gamma_n); \end{matrix} \right] g(s,t)dt\,ds$$

and $\Psi(x,y) = {}_1J^{a,\,b,\,b';\,c}_{x,\infty;y,\infty}\, f(x,y)$ *for* $c > 0$*, then*

$$\Psi(x,y) = \int_0^\infty \int_0^\infty s^{\lambda-1}\ t^{\mu-1}\ g(s,t)$$

$$ {}_1J^{a,\,b,\,b';\,c}_{x,\infty;y,\infty}\ x^{\lambda-1}y^{\mu-1}F^{p:q;r}_{\ell:m;n}\left[\begin{array}{c}(a_p):(b_q);(c_r);\\ \qquad sx,ty\\ (\alpha_\ell):(\beta_m);(\gamma_n);\end{array}\right] dt\,ds \tag{3.10}$$

provided that the double integrals involved exist.

Results similar to (3.9) and (3.10) hold for other operators (2.9), (2.13), (2.17), (2.20), (2.23) and (2.24) also from which similar results for other operators can easily be deduced as particular cases.

4 Results Analogues to Integration by Parts

Certain results analogues to integration by parts for the operators (2.1), (2.5), (2.9), (2.13), (2.17), (2.20), (2.23) and (2.24) are given in this section in the form of the following theorems:

Theorem 4.1. *For functions of two variables $f(x,y)$ and $g(x,y)$ defined in the positive quadrant of the xy-plane and $c>0$, we have*

$$\int_0^\infty \int_0^\infty f(xy)\,{}_1I^{a,\,b,\,b';\,c}_{0,x;0,y}\ g(x,y)dy\,dx$$
$$= \int_0^\infty \int_0^\infty g(xy)\,{}_1J^{a,\,b,\,b';\,c}_{x,\infty;y,\infty}\ f(x,y)dy\,dx \tag{4.1}$$

provided that each double integral exists.

Proof of Theorem 4.1 We have

$$\int_0^\infty \int_0^\infty f(x,y)\,{}_1I^{a,b,b';c}_{0,x;0,y}\ g(x,y)dy\,dx$$

$$= \int_0^\infty \int_0^\infty f(x,y)\frac{(xy)^{-a}}{\{\Gamma(c)\}^2}$$

$$\int_{u=0}^{u=x}\int_{v=0}^{v=y}(x-u)^{c-1}(y-v)^{c-1}\,F_1\left[\begin{array}{c}a,b,b';1-\frac{u}{x},1-\frac{v}{y}\\ c\ ;\end{array}\right] g(u,v)\,dvdu\,dydx$$

$$= \int_{u=0}^{\infty} \int_{v=0}^{\infty} \int_{x=u}^{\infty} \int_{y=v}^{\infty} \frac{(xy)^{-a}}{\{\Gamma(c)\}^2} f(x,y)(x-u)^{c-1}(y-v)^{c-1}$$

$$F_1 \left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c; \end{matrix} \right] g(u,v)\; dydx\; dvdu$$

$$= \int_{u=0}^{\infty} \int_{v=0}^{\infty} g(u,v) \frac{1}{\{\Gamma(c)\}^2}$$

$$\int_{x=u}^{\infty} \int_{y=v}^{\infty} (x-u)^{c-1}(y-v)^{c-1} F_1 \left[\begin{matrix} a,b,b';1-\frac{u}{x},1-\frac{v}{y} \\ c; \end{matrix} \right] x^{-a}\, y^{-a}\, f(x,y)\, dydx\, dvdu$$

$$= \int_{u=0}^{\infty} \int_{v=0}^{\infty} g(u,v)\, {}_1J_{u,\infty;v,\infty}^{a,b,b';c}\; f(u,v)dv\, du$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} g(x,y)\, {}_1J_{x,\infty;y,\infty}^{a,b,b';c}\; f(x,y)dy\, dx.$$

This completes the proof of Theorem 4.1. Similarly, we can prove theorems 4.2, 4.3 and 4.4.

Theorem 4.2. *For functions of two variables $f(x,y)$ and $g(x,y)$ defined in the positive quadrant of the xy-plane and $c > 0,\ c' > 0$, we have*

$$\begin{aligned} &\int_{0}^{\infty} \int_{0}^{\infty} f(x,y)\, {}_2I_{0,x;0,y}^{a,b,b';c,c'}\; g(x,y)dy\, dx \\ &= \int_{0}^{\infty} \int_{0}^{\infty} g(x,y)\, {}_2J_{x,\infty;y,\infty}^{a,b,b';c,c'}\; f(x,y)dy\, dx \end{aligned} \tag{4.2}$$

provided that each double integral exists.

Theorem 4.3. *Under the conditions stated in theorem 4.1, we have*

$$\begin{aligned} &\int_{0}^{\infty} \int_{0}^{\infty} f(x,y)\, {}_3I_{0,x;0,y}^{a,a',b,b';c}\; g(x,y)dy\, dx \\ &= \int_{0}^{\infty} \int_{0}^{\infty} g(x,y)\, {}_3J_{x,\infty;y,\infty}^{a,a',b,b';c}\; f(x,y)dy\, dx \end{aligned} \tag{4.3}$$

provided that each double integral exists.

Theorem 4.4. *Under the conditions stated in theorem 4.2, we have*

$$\begin{aligned} &\int_0^\infty \int_0^\infty f(x,y)\, {}_4I_{0,x;0,y}^{a,b;c,c'}\, g(x,y) dy\, dx \\ &= \int_0^\infty \int_0^\infty g(x,y)\, {}_4J_{x,\infty;y,\infty}^{a,b;c,c'}\, f(x,y) dy\, dx \end{aligned} \tag{4.4}$$

provided that each double integral exists.

From the theorems of this section certain interesting corollaries readily follow for the operators (2.2), (2.3), (2.4), (2.6), (2.7), (2.8), (2.10), (2.11), (2.14), (2.15), (2.16), (2.18), (2.19), (2.21) and (2.22).

References

[1] BUSCHMAN, R.G. (1964). *Fractional Integration.* Math. Japon. Vol. 9, pp. 99-106.

[2] BUSCHMAN, R.G. (1965). *Integrals of hypergeometric functions.* Math. Z., Vol. 89, pp. 74-76.

[3] ERDÉLYI, A. (1939). *Transformation of hypergeometric integrals by means of fractional integration by parts.* Quart. J. Math. Vol. 10, pp. 176-189.

[4] HARDY, G.H. AND LITTLEWOOD, J.E. (1928). *Some properties of fractional integrals, I.* Math. Z., Vol. 27, pp. 565-606.

[5] KHAN, M.A. (1974). *Certain Fractional q-Integrals and q-Derivatives.* Nanta Mathematica, Vol. VII, No.1, pp. 52-60.

[6] KHAN, M.A. (1992). *On Generalized Fractional q-Integrals.* Mathematica Balkanica, Vol. 6, Fasc. 2, pp. 155-163.

[7] KHAN, M.A. (1992). *Transformation of Generalized Fractional q-Integrals, I.* Mathematica Balkanica, Vol. 6, Fasc. 4, pp. 305-312.

[8] KHAN, M.A. (1993). *On certain transformations of Generalized fractional q-Integrals, II.* Mathematica Balkanica, Vol. 7, Fasc. 1, pp. 27-33.

[9] KOBER, H. (1940). *On Fractional integrals and derivatives.* Quart. J. Math. 11, pp. 193-211.

[10] LOVE, E. R. (1967). *Some integral equations involving hypergeometric functions.* Proc. Edinburgh Math. Soc. Vol. 15, pp. 169-198.

[11] RAINA, R.K. (1979). *On the Weyl fractional differentiation.* Pure. Appl. Math. Sci., Vol. 10, pp. 37-41.

[12] RAINA, R.K. (1985). *A note on the fractional derivatives of a general system of polynomials.* Indian J. Prue Appl., Math., Vol. 16, pp. 770-774.

[13] RAINA, R.K. (1986). *The Weyl fractional operator of a system of polynomials.* Rend. Sem. Mat. Univ. Padova, Vol. 76, pp. 171-176.

[14] ROSS, B. (ed.) (1975). *Fractional Calculus and its applications.* Lecture Notes in Mathematics, Vol. 457, Springer-Verlag (Berlín).

[15] SAIGO, M. (1978). *A remark on integral operators involving the Gauss hypergeometric functions.* Math. Rep. Kyushu Univ., Vol. 11, pp. 135-143.

[16] SAIGO, M. (1979). *A certain boundary value problem for the Euler-Darboux equation.* Math. Japon., Vol. 25, pp. 377-385.

[17] SAIGO, M. AND RAINA, R.K. (1988). *Fractional Calculus operators associated with a general class of polynomials.* Fukuoka Univ. Sci. Rep., Vol. 18, pp. 15-22.

[18] SRIVASTAVA, H.M. (1983). *The Weyl fractional integral of a general class of polynomials.* Boll. Un. Mat. Ital. B (6), Vol. 2, pp. 219-228.

[19] SRIVASTAVA, H.M. AND MANOCHA, H.L. (1984). *A treatise on generating functions.* Ellis Horwood, Chichester.

Mumtaz Ahmad Khan
Department of Applied Mathematics
Faculty of Engineering and Technology
Aligarh Muslim University,
Aligarh-202 002, U.P., India