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ELEMENTS FOR TEACHING GAME THEORY

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Abstract

This article is a lecture accepted for presentation and publication in the proceedings of the 2nd International Conference on the Teaching of Mathematics, held in Crete on July 2002. Some elements to have in mind for the teaching of Game Theory are given, using the author's teaching experience that relies on the viewpoint that encourages the formal development of mathematical concepts after proposing students problems related with these concepts and after stimulating and analyzing their intuitive approaches to the solution of these problems.

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1 Introduction

Game Theory should be included in the undergraduate programs of many majors, specially in those of economics, business administration, industrial engineering and, of course, of mathematics and statistics. It becomes indispensable in a globalized and technified society to become acquainted with theoretic points of view that help make decisions in conflict of interests situations. Game Theory gives a nice opportunity for university lecturers to carry out the essential role of stimulating the attitudes of observing, analyzing and theorizing in our future professionals, as a way to build a better world. Moreover, it is highly formative to know the basic results of a theory developed in the 20th century and to use the elements of probability to examine multiperson decision problems.

In the teaching-learning processes of mathematics, we should be careful about how and when to present the rigorous formalization of concepts and the use of specific techniques, since we must always bear in mind the importance of stimulating both an intuitive approach to the concepts that we are introducing and a creative use of the previous knowledge of our students. When we teach Game Theory we have a nice opportunity to apply these criteria through the collaborative learning and solving problems, according to the following sequence: understanding the problem (includes organization of the information and representation), intuitive approach to the solution, solution (or attempts of it) using previous knowledge, intuitive introduction of new concepts or theorems related with the problem, solution (or attempts of it) using the new concepts or theorems, formal and rigorous presentation of the new concepts or theorems, formal solution of the problem, search of other ways to solve it, explorations modifying the problem, creation of new problems and in depth study of the theoretical aspects using the intuition and the formalization. With this didactical propose, I made it easy for my students to understand the concepts of Game Theory, specially Nash equilibrium and mixed strategies for non zero-sum games and their applications.

A fundamental task of teachers of any subject, but specially of mathematics, is guiding their students learn to learn, and helping them get self-confidence on their learning capabilities. Game theory is specially favorable for the performance of this task, because it deals with topics related with our daily life, which are becoming more important: situations in which there are conflict of interests, in which it is necessary to decide looking for the most suitable choice and considering what the other persons, with similar interests, may do. It is very good for the motivation to be aware that these situations happen not only in parlor games, but also in games in a wider sense, which we play or whose play we see day to day: driving a car in a big city, trading the price of a commodity (as buyer or as seller), advertising, defending or accusing a prisoner, proposing a salary, designing economic policies in a country, facing a war, etc. All this favors the motivation and contributes to the presentation and development of the concepts starting from problems and making dynamical and collaborative classes with intuitive approaches prior to the formalizations proper of the theory. The cases of noncooperative games with two players and a finite number of strategies are particularly interesting because the students, appropriately guided in using their intuition and with the aid of relatively elementary mathematics, usually arrive at solutions or criterions that are in fact part of the theory, even though not yet formalized. When the students verify this, they strengthen in self-security about their learning capabilities.

Regarding intuition and mathematics, it is appropriate to recall what Efraim Fischbein wrote in his book *Intuition in science and mathematics.* He does not believe intuitive reasoning to be present in certain stages of the development of intelligence only, but instead that typically intuitive forces guide the way we solve problems and carry out interpretations, no matter how old -or young- we are. Furthermore, even when faced with highly abstract concepts, we tend - almost automatically- to represent them in a way that makes them intuitively accessible. However, we must bear in mind that this same author warns that "by exaggerating the role of intuitive prompts, one run the risk of hiding the genuine mathematical content instead of revealing it. By resorting too early to a 'purified', strictly deductive version of a certain mathematical domain, one runs the risk of stifling the student's personal mathematical reasoning instead of developing it". (Fishbein 1987, p.214)

The present article is meant to show a way of working with basic aspects of Game Theory, which agrees with the outline of the previous paragraphs.

2 Playing in the Classroom

Students are divided into two groups: Alpha and Beta. From each group two students are selected to be the players ($P1$ and $P2$) of games whose rules are to be announced. So that in each group there is a $P1$ and a $P2$. The idea is to obtain results in the separate groups for later comparison. Each player calls from his group a team of "advisers" that will help him choose the best decision. Neither players nor different teams are allowed to communicate, and the decision must be rational.

Game 1

For this game I give each player two cards, named $C1$ and $C2$. Each card holds a written demand that I will fulfill:

C1: Give the other player 3 dollars.

C2: Give me 1 dollar.

Each player must choose one card only, and give it back to me. So they must decide which card to choose in order for them to get the greatest possible benefit from their participation in the game.

After a prudential time for discussion with their advisers, players from both groups turn in one card each. After reading them, I fulfill each card's demand.[1]

Understanding the problem is a fundamental stage and generally, after some time for group deliberation, the information is organized in one of the following forms:

- *Lists of payoffs*

Payoffs to P1:

P1's Choice	P2's Choice	Payoff to P1
$C1$	$C1$	3
$C1$	$C2$	0
$C2$	$C1$	4
$C2$	$C2$	1

Payoffs to P2:

P1's Choice	P2's Choice	Payoff to P2
$C1$	$C1$	3
$C1$	$C2$	4
$C2$	$C1$	0
$C2$	$C2$	1

[1]This game is based on Aumann's version of the known game "prisoner's dilemma"

- *Matrix tables*

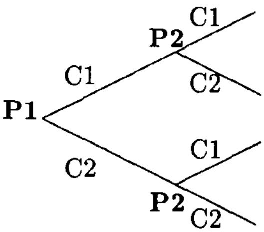
Payoffs to P1:

		P2 C1	P2 C2
$P1$	C1	3	0
	C2	4	1

Payoffs to P2:

	P2 C1	P2 C2
C1	3	4
C2	0	1

- *Trees*



Payoffs to **P1**	Payoffs to **P2**
3	3
0	4
4	0
1	1

It is of great stimulus for the student's learning to learn capacities to realize later that, without consciously knowing it, they had been using concepts and representations that are common use in Game Theory. Thus, their way of organizing the information by means of "payoff lists" corresponds to the *payoff functions* of the proposed game, and the two other ways are just the two mayor representations for describing games: the *normal form* and the *extensive form*, respectively. It is then a very simple task to resume the two matrix tables in a bimatrix table, just as the ones used for the analysis of normal form games.

		$P2$ C1	$P2$ C2
$P1$	C1	(3,3)	(0,4)
	C2	(4,0)	(1,1)

It is generally the case that in both groups, Alpha and Beta, players use the $C2$ option. When they are asked to explain the rationale behind their choice, they do it by means of the scheme they used to organize information and by certain criteria that are in fact an intuitive approximation to the notion of *strict domination of strategies.* It is clear that even when apparently they would be better off using both $C1$, rationality (and a certain sense of self-assurance) forces them to choose $C2$. Next they are asked to relate this game to similar real-life situations. In one occasion a group showed that the same situation could be observed in an arms race between two countries: both are conscious of the convenience of decreasing their expenses in weapon systems, but neither will risk to do so without being reasonable sure that the other also would. As a result of distrust, they continue expensing enormous amounts of money in weapons.

We continue posing two new problems; both already resumed in their bimatrix form:

Game 2

$P2$

$P1$

	Red	Yellow	Green
White	(4,3)	(3,4)	(4,5)
Black	(0,6)	(5,0)	(3,4)

Game 3

$P2$

$P1$

	Red	Yellow	Green
White	(1,9)	(3,4)	(3,8)
Black	(2,4)	(0,4)	(4,6)
Brown	(3,5)	(2,6)	(3,4)

Working in groups as before, I give the students time enough to study the problems. By using the notion of strictly dominated strategies, but without any further formalization, they find the solution for *Game 2*: P1 chooses White and P2 chooses Green, and the players receive the payoffs 4 and 5, respectively. Through this problem students learn to work with the rationality of Game Theory; they realize that at first P1 has no strictly dominated strategy, but that on the other hand Yellow is strictly dominated by Green for P2, so this starts their process of finding a solution.

Game 3 brings a particular difficulty: neither player has a strictly dominated strategy. However, students generally come to the solution that corresponds to a Nash equilibrium: the best choice for P1 is "Black" and the best one for P2 is "Green". Difficulties they find to explain how they came to such a solution, added to the lack of formal algorithms, make us think that their solution is purely intuitive. The fact of receiving the teacher's -and the whole class's- approval of their solution reinforces their self-security; the next task is to find a rational way to arrive at the solution. This is a crucial part of the learning process of Game Theory, since the search for a more careful description of the player's rationality is in turn the beginning of an understanding of the rationality behind this theory. At this stage they are not yet informed of formal definitions or techniques, which when given from the beginning lead to a purely deductive learning, and sometimes to a merely mechanical application of techniques, shortening so this important phase of intuitive and creative approach. It takes a little bit of time, but it is generally the case that after a period of discussion within the groups, and between groups, students grasp the idea of thinking what a player would do if he knew the other player's choice in advance. So they start ticking the "most convenient" payoffs in each case, and the solution is then determined by the strategies that correspond to a box having both components of the pair of payoffs ticked. After this experience, it is clear for the students that the absence of strictly dominated strategies does not imply the absence of a solution, and it is interesting to ask them to attempt a definition of the concept of "rational solution", which in the theory corresponds to Nash equilibrium. The students clearly perceive the necessity of formalization, and they are asked to take care of it. Regarding this stage, I had an excellent experience when receiving the following explanation, as an attempt to "define a Nash equilibrium" for games similar to the given ones: Two lists are made:

If P2 chose	*then P1 would choose*
Red	Brown
Yellow	White
Green	Black

If P1 chose	*then P2 would choose*
White	Yellow
Black	Green
Brown	Red

Since Green - Black is in the first list and Black - Green is in the second, this pair of strategies is the rational solution for the game. These lists are in fact the best-response correspondences for the players; so essentially the definition is that of Nash equilibrium in pure strategies in terms of the best-response correspondences that are commonly given for finite two-person games.[2]

3 Creating Games

An activity that is frequently given little importance is that of creating problems. This task should parallel that of solving problems, since it stimulates creativity, helps to fix ideas and concepts that are being introduced, and presents new difficulties that require the introduction of new concepts or techniques in order for them to be overcome. It is very attractive and motivating for the students to attempt to get through with the difficulties created by themselves; specially when they are conscious of the criteria they should use, but they find them insufficient. When asked to create games similar to those ones they were faced to, students easily come with games having more than one Nash equilibrium, games in which a player's best response to a certain strategy from his opponent is not unique (this is taken to introduce the concept of correspondence, rather than that of function); and -more interesting- games that have no Nash equilibrium according to the given criterion. After discussing some selected problems, formal definitions of game, payoff function, strictly dominated strategy, best-response correspondence and Nash equilibrium are presented for two-person games. The equivalence of the definitions of Nash equilibrium in terms of the best-response correspondence and of the payoff functions is highlighted. By observing a bimatrix game with a Nash equilibrium, they verify that being (s,t) an equilibrium point, if player 1 changes his strategy while player 2 does not, then the payoff received by the former is never as good as that he would receive in (s,t). A symmetric verification is made for the case of player 2: if he deviated from his equilibrium strategy while player 1 did not, then his payoff would never increase. After that, the formal statement of Nash theorem is presented: every finite game (a game with

[2] If R_1 and R_2 are correspondences defining the sets of players' best response to each other's strategy, the pair (s,t) is a Nash equilibrium if and only if $s \in R_1(t)$ and $t \in R_2(s)$

a finite number of players, each one having a finite number of strategies only) there is a Nash equilibrium.

Here is a selection of games, taken from those presented by the students:

Game (a)

	S	T	U
A	(3,6)	(7,1)	(2,6)
B	(4,1)	(7,5)	(5,8)

Game (b)

	S	T
A	(2,4)	(3,9)
B	(5,3)	(2,1)

Game (c)

	S	T	U	V
A	(2,4)	(3,9)	(7,1)	(7,0)
B	(5,3)	(2,1)	(6,4)	(4,1)
C	(0,5)	(4,3)	(3,3)	(9,2)

Game (d)

	S	T
A	(2,-2)	(-6,6)
B	(-3,3)	(4,-4)

In the four of them students use the technique of underlining the payoffs that correspond to a player's best response, when considering that his opponent uses some fixed strategy.

- *In Game (a)* it is easily seen that player 1 is indifferent between choosing his strategies A or B if he knew that player 2 will choose T. Similarly, player 2 is indifferent between S and U, as long as he is certain that player 1 will choose A. Using the best-response correspondences, we have:

  $$R_1(S) = \{B\}; R_1(T) = \{A, B\}; R_1(U) = \{B\}$$

  $$R_2(A) = \{S, U\}; R_2(B) = \{U\}$$

  It is a simple matter to see that the pair (B, U) is a Nash equilibrium, and we can find this point either by the elimination of strictly dominated strategies, or by observing that $B \in R_1(U)$ and at the same time $U \in R_2(B)$.

- *In Game (b)* two Nash equilibria are obtained. This fact causes controversy on which one should be used, and motivates commentaries on the interchangeability and equivalence of equilibria, as well as on the idea of subgame perfect equilibrium. Furthermore, when the concept of mixed strategy was introduced later, it was very interesting that they found out their proposed games had a third Nash equilibrium.

- *In Game (c)*, formed from *Game (b)* by adding strategies to both players, no Nash equilibria could be obtained. Expectative and doubt arose among students, since it was natural for them to think that a counter-example had been found for the Nash theorem, stated before. Then they were suggested to look for more simple games having this property. That is how *Game (d)* came into scene; the latter has also another interesting particularity: it is a zero-sum game, that is, a game in which the amount obtained by a player is the amount lost by the other.

- *In Game (d)*, in the absence of strictly dominated strategies, and being unable to find a clear criterion to guide the players' choice, I suggested them to think that the players have actually more than two ways to carry out their choice. In most cases, students found, as a third way to "choose" an alternative, a random device: tossing a coin. At this point the natural question is: why not to use a die instead of a coin? Or why not a roulette? Thus, for instance, player 1 could choose between A or B by tossing coin: if it comes up heads he chooses A and if it comes up tails he chooses B; and player 2 has the possibility of choosing between S or T by throwing a die: if the outcome is 1 he chooses S, while if the outcome is 2,3,4,5 or 6, he chooses T. It is clear that when using a die to "choose" between two alternatives, many different assignments could be done between numbers and strategies. A question is now in order: are these random devices the most convenient? Were the students to accept that random devices are indeed necessary, the formalization suggests the use of probabilities and expectation. With the aid of these tools, the students themselves redefine in a natural way the (expected) payoff for each player, and it is interesting to guide them towards an extension of the definition of Nash equilibrium, by asking them to compute and compare some expected payoffs. For instance, in *Game (d)*, assuming that players carry out their choices by tossing a coin and throwing a die, respectively, and thinking of the correspondence between outcomes and strategies given above, this means that player 1 chooses A with probability $\frac{1}{2}$ and B with probability $\frac{1}{2}$ as well; while player 2 chooses S with probability $\frac{1}{6}$ and T with probability $\frac{5}{6}$.

The expected payoff for player 1 corresponding to these probabilities, which we may call $EP_1((\frac{1}{2},\frac{1}{2})(\frac{1}{6},\frac{5}{6}))$, or simply $EP_1(\frac{1}{2},\frac{1}{6})$, can be obtained from the matrix of payoffs for player 1, in which the probabilities are written too:

		1/6	5/6
		S	T
1/2	A	2	-6
1/2	B	-3	4

$$EP_1(1/2,1/6) = 2\times\frac{1}{2}\times\frac{1}{6}-6\times\frac{1}{2}\times\frac{5}{6}-3\times\frac{1}{2}\times\frac{1}{6}+4\times\frac{1}{2}\times\frac{5}{6} = -\frac{11}{12}$$

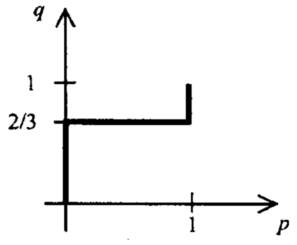
With a similar computation we obtain $EP_2(1/2,1/6) = \frac{11}{12}$. However, this random device to choose their strategies is not the most convenient for any of them. To see this we can consider, for instance, that player 1 decides to use a die instead of a coin while player 2 maintains his previous device. In this case, assigning a probability of 1/6 to A and 5/6 to B, we would obtain $EP_1(1/6,1/6) = 2\times\frac{1}{6}\times\frac{1}{6}-6\times\frac{1}{6}\times\frac{5}{6}-3\times\frac{5}{6}\times\frac{1}{6}+4\times\frac{5}{6}\times\frac{5}{6} = \frac{19}{12}$, which means that player 1 has improved his expected payoff. The moral is that $(1/2,1/2)$ for player 1, and $(1/6,5/6)$ for player 2 cannot be a Nash equilibrium. The search for the most convenient device for choosing at random a strategy makes them think of the most convenient probability that should be assigned to each strategy. With a little help they come to realize that the best "practical" device is neither a coin nor a die, but something like a two-color roulette, with the portion covered by each color being proportional to the assigned probabilities. Thus, for instance, player 1 could use a roulette having 3/5 of its area painted in Green and 2/5 in Blue; if the roulette stops in Green he chooses A, if it stops in Blue he chooses B. After these experiences it is natural to extend the set of strategies for each player, calling *pure strategies* the original strategies they had been working with, and introducing the concept of *mixed strategies* as probability assignments over the pure ones. Restricting our work to two-person games with only two pure strategies for each player, and recalling the best-response criterion used to define the concept of Nash equilibrium in pure strategies, we look at the general expression for the expected payoff for each player and plot the best-response correspondences, next we intuitively conclude that the points where these two curves intersect determine all Nash equilibria, including pure strategy equilibria, if any. Furthermore, looking at the graphics we can figure out that in two-person games with only two strategies for each one, there will always be at least one Nash equilibrium. In the case of *Game (d)*, assigning probabilities p and $(1-p)$ to player 1's pure strategies A and B, respectively; and probabilities q and $(1-q)$ to player 2's strategies S and T, respectively, we obtain:

$$EP_1(p,q) = 15pq - 10p - 7q + 4 = p(15q - 10) - 7q + 4.$$

Since p and q can only take values in the interval $[0,1]$, and since this function is linear in p, it can be seen that player 1's best response to values of q that make the expression $15q-10$ positive (i.e., $q \in]2/3,1]$) is choosing the greatest possible value for p, that is $p=1$. Analogously, his best response to values of q that turn the expression $15q-10$ negative (i.e., $q \in [0,2/3[$) is choosing the least possible value for p, that is $p=0$. If $q = 2/3$, the expression $15q-10$ vanishes and the expected payoff for player 1 no longer depends on the value he chooses for p; in consequence, p can take any value in the interval $[0,1]$. To resume, player 1's best response to the mixed strategy $(q, 1-q)$ of player 2, which we call $R_1(q)$ for short, is

$$R_1(q) = \begin{cases} \{1\} & if \quad q \in]2/3,1] \\ \{0\} & if \quad q \in [0,2/3[\\ [0,1] & if \quad q = 2/3 \end{cases}$$

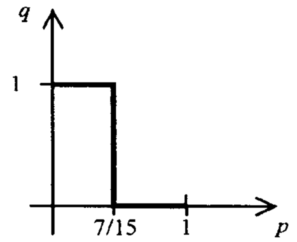
and graphically:



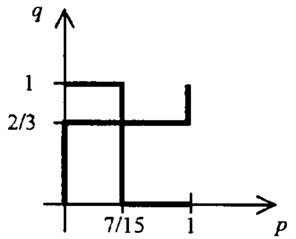
With a similar reasoning, we obtain, $EP_2(p,q) = q(7-15p) + 10p - 4$, and from this

$$R_2(p) = \begin{cases} \{1\} & if \quad p \in \ [0, 7/15[\\ \{0\} & if \quad p \in \ ]7/15, 1] \\ [0,1] & if \quad p = \ 7/15 \end{cases}$$

and graphically:



When plotted in the same coordinate system, the intersection of these two graphs gives us, for each player, a mixed strategy that is the best response to his opponent's choice. Thus, we see that the only Nash equilibrium is the pair of mixed strategies $((7/15, 8/15), (2/3, 1/3))$.



This visualization of Nash equilibria is a very interesting tool for the analysis, creation of problems and the stimulus of research. It is very important to induce the students to make conjectures on the existence of Nash equilibria and on the greatest possible number of these, as well as having them design their own examples and counter-examples to support or discard their conjectures. We can thus obtain a whole rank of cases, from the "intuitive security" of the existence of at least one Nash equilibrium, up to the design of games with infinitely many equilibrium points.

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