THREE VARIABLE ANALOGUE OF BOAS AND BUCK TYPE GENERATING FUNCTIONS AND ITS GENERALIZATIONS TO m-VARIABLES

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Abstract

The present papers deals with three variable analogue of Boas and Buck [14] type generating functions for polynomials of two variables and then the same has been extended for m-variable analogue. The results obtained are extensions of those obtained by us in our earlier paper [14].

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1. Introduction

Extending the results of Boas and Buck [14] the present authors [14] considered two-variable analogues of certain theorems given by Boas and Buck. Let $P_n(x,y)$ be a polynomial in two variables defined by means of the generating functions of the form

$$A(t)\phi(xH(t))\psi(yH(t)) = \sum_{n=0}^{\infty} P_n(x,y)t^n \tag{1.1}$$

and

$$\phi(t) = \sum_{n=0}^{\infty} \gamma_n \, t^n, \qquad \gamma_0 \neq 0 \tag{1.2}$$

$$\psi(t) = \sum_{n=0}^{\infty} \delta_n \, t^n, \qquad \delta_0 \neq 0 \tag{1.3}$$

$$A(t) = \sum_{n=0}^{\infty} a_n \, t^n, \qquad a_0 \neq 0 \tag{1.4}$$

$$H(t) = \sum_{n=0}^{\infty} h_n \, t^{n+\frac{1}{2}}, \; h_0 \neq 0 \tag{1.5}$$

Then the following theorems analogous to those obtained by Boas and Buck [14] hold for two variable polynomials:

Theorem A. If $P_n(x,y)$ is defined by (1.1) with (1.2), (1.3), (1.4) and (1.5) holding, $P_n(x,y)$ is a polynomial in x and y and $P_n(x,y)$ is of degree precisely n if and only if $\gamma_n \neq 0$ and $\delta_n \neq 0$.

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Theorem B. For the polynomials $P_n(x, y)$ defined by (1.1) with (1.2), (1.3), (1.4) and (1.5) holding, and $\gamma_n \neq 0$, $\delta_n \neq 0$, there exist sequences of numbers α_k and β_k such that, for $n \geq 1$,

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) P_n(x, y) - n\ P_n(x, y)$$

$$= -\sum_{k=0}^{n-1} \alpha_k\ P_{n-1-k}(x, y) - \sum_{k=0}^{n-1} \beta_k \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) P_{n-1-k}(x, y) \tag{1.6}$$

Indeed,

$$\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^{n+1} \tag{1.7}$$

$$\frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \tag{1.8}$$

In our earlier paper [14], a function extension was given by considering the generating relation

$$A(t)\phi(xH(t) + g(t))\phi(yH(t) + r(t)) = \sum_{n=0}^{\infty} f_n(x, y)t^n \tag{1.9}$$

In which

$$\phi(t) = \sum_{n=0}^{\infty} \gamma_n\ t^n, \qquad \gamma_0 \neq 0 \tag{1.10}$$

$$\psi(t) = \sum_{n=0}^{\infty} \delta_n \ t^n, \qquad \delta_0 \ \neq \ 0 \tag{1.11}$$

$$A(t) = \sum_{n=0}^{\infty} a_n \ t^n, \qquad a_0 \ \neq \ 0 \tag{1.12}$$

$$H(t) = \sum_{n=0}^{\infty} h_n \ t^{n+\frac{1}{2}}, \ h_0 \ \neq \ 0 \tag{1.13}$$

$$g(t) = \sum_{n=0}^{\infty} g_n \ t^{n+2} \tag{1.14}$$

and

$$r(t) = \sum_{n=0}^{\infty} r_n \ t^{n+2} \tag{1.15}$$

The following theorems were proved to hold:

Theorem C. If $P_n(x, y)$ is defined by (1.9) with (1.10), (1.11), (1.12), (1.13), (1.14) and (1.15) holding, $f_n(x, y)$ is a polynomial in x and y and $f_n(x, y)$ is of degree precisely n if and only if $\gamma_n \ \neq \ 0$ and $\delta_n \ \neq \ 0$.

Theorem D. For the polynomials $f_n(x, y)$ defined by (1.9) with (1.10), (1.11), (1.12), (1.13), (1.14) and (1.15) holding, and $\gamma_n \ \neq \ 0$, $\delta_n \ \neq \ 0$, there exist sequences of numbers α_k, β_k, λ_k and μ_k such that, for $n \ \geq \ 1$

$$\begin{aligned}
&\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f_n(x, y) - n \ f_n(x, y) \\
&= -\sum_{k=0}^{n-1} \alpha_k \ f_{n-1-k}(x, y) - \sum_{k=0}^{n-1} \beta_k \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \ f_{n-1-k}(x, y) \\
&= -\sum_{k=0}^{n-1} \left(\lambda_k\frac{\partial}{\partial x} + \mu_k\frac{\partial}{\partial y}\right) f_{n-1-k}(x, y)
\end{aligned} \tag{1.16}$$

Indeed,

$$\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^{n+1} \tag{1.17}$$

$$\frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \tag{1.18}$$

$$\frac{tg'(t)}{H(t)} = \sum_{n=0}^{\infty} \lambda_n t^{n+1} \tag{1.19}$$

$$\frac{tr'(t)}{H(t)} = \sum_{n=0}^{\infty} \mu_n t^{n+1} \tag{1.20}$$

2. Main Results

Here we obtain three variable analogues of theorems A, B, C and D mentioned above. Let $P_n(x_1, x_2, x_3)$ be a polynomial in three variables defined by means of the generating functions of the form

$$A(t)\ \phi_1(x_1 H(t))\ \ \phi_2(x_2 H(t))\ \phi_3(x_3 H(t)) = \sum_{n=0}^{\infty} P_n(x_1, x_2, x_3) t^n \tag{2.1}$$

and

$$\phi_1(t) = \sum_{n=0}^{\infty} \gamma_n\ t^n, \quad \gamma_0\ \neq\ 0 \tag{2.2}$$

$$\phi_2(t) = \sum_{n=0}^{\infty} \delta_n\ t^n, \quad \delta_0\ \neq\ 0 \tag{2.3}$$

$$\phi_3(t) = \sum_{n=0}^{\infty} \lambda_n\ t^n, \quad \lambda_0\ \neq\ 0 \tag{2.4}$$

$$A(t) = \sum_{n=0}^{\infty} a_n \; t^n, \quad a_0 \; \neq \; 0 \tag{2.5}$$

$$H(t) = \sum_{n=0}^{\infty} h_n t^{n+\frac{1}{2}}, h_0 \; \neq \; 0 \tag{2.6}$$

Then the following theorem holds:

Theorem 1. If $P_n(x_1, x_2, x_3)$ in defined by (2.1), with (2.2), (2.3), (2.4), (2.5) and (2.6) holding, $P_n(x_1, x_2, x_3)$ is a polynomial in x_1, x_2 and x_3 and $P_n(x_1, x_2, x_3)$ is of degree precisely n if and only if $\gamma_n \neq 0$, $\delta_n \neq 0$ and $\lambda_n \neq 0$.

Proof: Let

$$P_n(x_1, x_2, x_3) = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} S(k, r, s, n) \; x_1^k \; x_2^r \; x_3^s \tag{2.7}$$

Then

$$A(t) \; \phi_1(x_1 H(t)) \; \phi_2(x_2 H(t)) \; \phi_3(x_3 H(t))$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} S(k, r, s, n) \; x_1^k \; x_2^r \; x_3^s \; t^n$$

so that m partial differentiations with respect to x_1, followed by putting $x_1 = 0$, yields

$$A(t) \; [H(t)]^m \; \phi_1^{(m)}(0) \; \phi_2(x_2 H(t)) \; \phi_3(x_3 H(t))$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (m!) \; S(m, r, s, n) \;\; x_2^r \; x_3^s \; t^n \tag{2.8}$$

Similarly, m partial differentiation of (2.8) with respect to x_2, followed by putting $x_2 = 0$, gives

$$A(t)\ [H(t)]^{2m}\ \phi_1^{(m)}(0)\ \phi_2^{(m)}(0)\ \phi_3(x_3 H(t))$$

$$= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\ (m!)^2\ S(m,m,s,n)\ x_3^s\ t^n \tag{2.9}$$

Again, m partial differentiation of (2.9) with respect to x_3, followed by putting $x_3 = 0$, gives

$$A(t)\ [H(t)]^{3m}\ \phi_1^{(m)}(0)\ \phi_2^{(m)}(0)\ \phi_3^{(m)}(0) = \sum_{n=0}^{\infty}\ (m!)^3\ S(m,m,m,n)\ t^n \tag{2.10}$$

Because of (2.2)-(2.6), one can write (2.10) as

$$A(t)\ [H(t)]^{3m}\ \phi_1^{(m)}(0)\ \phi_2^{(m)}(0)\ \phi_3^{(m)}(0)$$

$$= a_0\ h_0^{3m}\ t^m\ \gamma_m\ \delta_m\ \lambda_m\ (m!)^3 + \sum_{n=m+1}^{\infty}\ C(m,m,m,n)\ t^n, \tag{2.11}$$

In which the precise nature of $C(m,m,m,n)$ is not important to us.

Comparison of (2.10) and (2.11) leads to

$$s(m,m,m,n) = 0 \qquad for\ n\ <\ m \tag{2.12}$$

$$s(m,m,m,n) = a_0\ h_0^{3m}\ \gamma_m\ \delta_m \lambda_m \tag{2.13}$$

The condition (2.12) shows that $P_n(x_1, x_2, x_3)$ is a polynomial of degree $\leq n$. The condition (2.13) with $3m$ replace by n, shows that $P_n(x_1, x_2, x_3)$ is of degree precisely n if and only if $\gamma_n\ \neq\ 0$, $\delta_n\ \neq\ 0$, $\lambda_n\ \neq\ 0$, since $a_0 h_0\ \neq\ 0$ by (2.5) and (2.6).

Theorem 2. For polynomials $P_n(x_1, x_2, x_3)$ defined by (2.1), with (2.2), (2.3), (2.4), (2.5) and (2.6) holding, and $\gamma_n \neq 0$, $\delta_n \neq 0$, $\lambda_n \neq 0$, there exist sequences of numbers α_k and β_k such that, for $n \geq 1$

$$\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right) P_n(x_1, x_2, x_3) - nP_n(x_1, x_2, x_3)$$

$$= -\sum_{k=0}^{n-1} \alpha_k P_{n-1-k}(x_1, x_2, x_3)$$

$$-\sum_{k=0}^{n-1} \beta_k \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right) P_{n-1-k}(x_1, x_2, x_3) \tag{2.14}$$

Indeed,

$$\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^{n+1} \tag{2.15}$$

$$\frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \tag{2.16}$$

Proof: Let

$$F = A(t)\ \phi_1(x_1 H(t))\ \phi_2(x_2 H(t))\ \phi_3(x_3 H(t)) \tag{2.17}$$

Then

$$\frac{\partial F}{\partial x_1} = H(t)A(t)\ \phi_1'\ \phi_2\ \phi_3 \tag{2.18}$$

$$\frac{\partial F}{\partial x_2} = H(t)A(t)\ \phi_1\ \phi_2'\ \phi_3 \tag{2.19}$$

$$\frac{\partial F}{\partial x_3} = H(t)A(t)\ \phi_1\ \phi_2\ \phi_3' \tag{2.20}$$

$$\frac{\partial F}{\partial t} = A'(t)\ \phi_1\ \phi_2\ \phi_3$$

$$+x_1 H'(t) A(t)\ \phi_1'\ \phi_2\ \phi_3 + x_2 H'(t) A(t)\ \phi_1\ \phi_2'\ \phi_3 + x_3 H'(t) A(t)\ \phi_1\ \phi_2\ \phi_3' \tag{2.21}$$

Eliminating ϕ_1, ϕ_1', ϕ_2, ϕ_2', ϕ_3 and ϕ_3' with the aid of (2.17), (2.18), (2.19), (2.20) and (2.21), the result may be written in the form

$$\frac{tH'(t)}{H(t)}\left[x_1\frac{\partial F}{\partial x_1} + x_2\frac{\partial F}{\partial x_2} + x_3\frac{\partial F}{\partial x_3}\right] - t\frac{\partial F}{\partial t} = -\frac{tA'(t)}{A(t)}F \tag{2.22}$$

If we define α_n and β_n by (2.15) and (2.16) and recall that

$$F = \sum_{n=0}^{\infty} P_n(x_1, x_2, x_3) t^n$$

equation (2.22) leads us to

$$\left[1 + \sum_{n=0}^{\infty} \beta_n t^{n+1}\right]\left[\sum_{n=0}^{\infty}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right) P_n(x_1, x_2, x_3) t^n\right]$$

$$-\sum_{n=0}^{\infty} nP_n(x_1, x_2, x_3) t^n = -\left[\sum_{n=0}^{\infty}\alpha_n t^{n+1}\right]\left[\sum_{n=0}^{\infty} P_n(x_1, x_2, x_3) t^n\right]$$

or

$$\sum_{n=0}^{\infty}\left[\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right) P_n(x_1, x_2, x_3) - nP_n(x_1, x_2, x_3)\right] t^n$$

$$= -\sum_{n=0}^{\infty}\sum_{k=0}^{n} \alpha_k P_{n-k}(x_1, x_2, x_3) t^{n+1}$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{n} \beta_k \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) P_{n-k}(x_1, x_2, x_3) t^{n+1}$$

$$= -\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \alpha_k P_{n-1-k}(x_1, x_2, x_3) t^{n}$$

$$-\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \beta_k \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) P_{n-1-k}(x_1, x_2, x_3) t^{n} \quad (2.23)$$

from which the result given below follows at once:

$$\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) P_n(x_1, x_2, x_3) - nP_n(x_1, x_2, x_3)$$

$$= -\sum_{k=0}^{n-1} \alpha_k P_{n-1-k}(x_1, x_2, x_3)$$

$$-\sum_{k=0}^{n-1} \beta_k \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) P_{n-1-k}(x_1, x_2, x_3) \quad (2.24)$$

It is important that the α_k and β_k in (2.24) are independent of n.

Example: Consider the polynomials $f_n(x_1, x_2, x_3)$ of [13] in which

$$(1-t)^{-c}\, \phi_1 \left[\frac{-4x_1 t}{(1-t)^2} \right] \, \phi_2 \left[\frac{-4x_2 t}{(1-t)^2} \right] \, \phi_3 \left[\frac{-4x_3 t}{(1-t)^2} \right] = \sum_{n=0}^{\infty} f_n(x_1, x_2, x_3) t^n \quad (2.25)$$

The $f_n(x_1, x_2, x_3)$ fit into the Boas & Buck theory with

$$A(t) = (1-t)^{-c}, \qquad H(t) = \frac{-4t}{(1-t)^2}$$

$$\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} ct^{n+1}, \quad \frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} 2t^{n+1}$$

Hence $\alpha_n = c$, $\beta_n = 2$ and the relation (2.24) becomes

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right) P_n(x_1, x_2, x_3) - nP_n(x_1, x_2, x_3)$$

$$= -c\sum_{k=0}^{n-1} f_{n-1-k}(x_1, x_2, x_3)$$

$$- 2\sum_{k=0}^{n-1}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right) f_{n-1-k}(x_1, x_2, x_3) \qquad (2.26)$$

which is equation (3.6) of theorem of [13] with the right member written in reverse order.

The Boas and Buck type work obtained in the present paper for three variable polynomials applies to polynomials considered in [11] and [13] but not to those of [9].

3. An Extension

Consider the generating relation

$$A(t)\phi_1[x_1H(t) + g(t)]\ \phi_2[x_2H(t) + q(t)]\phi_3[x_3H(t) + r(t)]$$

$$= \sum_{n=0}^{\infty} f_n(x_1, x_2, x_3)t^n \qquad (3.1)$$

In which

$$\phi_1(t) = \sum_{n=0}^{\infty} \gamma_n\ t^n, \qquad \gamma_0\ \neq\ 0 \qquad (3.2)$$

$$\phi_2(t) = \sum_{n=0}^{\infty} \delta_n \ t^n, \qquad \delta_0 \ \neq \ 0 \tag{3.3}$$

$$\phi_3(t) = \sum_{n=0}^{\infty} \lambda_n \ t^n, \qquad \lambda_0 \ \neq \ 0 \tag{3.4}$$

$$A(t) = \sum_{n=0}^{\infty} a_n \ t^n, \qquad a_0 \ \neq \ 0 \tag{3.5}$$

$$H(t) = \sum_{n=0}^{\infty} h_n \ t^{n+\frac{1}{3}}, \ h_0 \ \neq \ 0 \tag{3.6}$$

$$g(t) = \sum_{n=0}^{\infty} g_n \ t^{n+2} \tag{3.7}$$

$$q(t) = \sum_{n=0}^{\infty} q_n \ t^{n+2} \tag{3.8}$$

and

$$r(t) = \sum_{n=0}^{\infty} r_n \ t^{n+2} \tag{3.9}$$

Note that $g(t)$, $q(t)$ and $r(t)$ are permitted to be identically zero. It is not necessary to require that $g'(0) = 0$, $q'(0) = 0$ and $r'(0) = 0$, but these involve no loss of generality.

Theorem 3. If $f_n(x_1, x_2, x_3)$ is defined by (3.1) with (3.2)-(3.9) holding, $f_n(x_1, x_2, x_3)$ is a polynomial in x_1, x_2 and x_3, and $f_n(x_1, x_2, x_3)$ is of degree precisely n if and only if $\gamma_n \neq 0$, $\delta_n \neq 0$ and $\lambda_n \neq 0$.

Proof: The Proof is similar to that of theorem 1. Put

$$f_n(x_1, x_2, x_3) = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} s(k, r, j, n)x_1^k\ x_2^r\ x_3^j \tag{3.10}$$

Then

$$A(t)\phi_1[x_1H(t)+g(t)]\ \phi_2[x_2H(t)+q(t)]\phi_3[x_3H(t)+r(t)]$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} s(k, r, j, n)x_1^k\ x_2^r\ x_3^j t^n,$$

from which m times partial differentiation with respect to x_1 followed by putting $x_1 = 0$ and then m times partial differentiation with respect to x_2 followed by putting $x_2 = 0$ and finally m times partial differentiation with respect to x_3 followed by putting $x_3 = 0$, yields

$$A(t)[H(t)]^{3m}\ \phi_1^{(m)}[g(t)]\ \phi_2^{(m)}[q(t)]\ \phi_3^{(m)}[r(t)] = \sum_{n=0}^{\infty}(m!)^3 s(m, m, m, n)t^n \tag{3.11}$$

Because of (3.2)-(3.9), we obtain

$$A(t)[H(t)]^{3m}\ \phi_1^{(m)}[g(t)]\ \phi_2^{(m)}[q(t)]\ \phi_3^{(m)}[r(t)]$$

$$= a_0h_0^{3m}(m!)^3\gamma_m\delta_m\lambda_m t^m = \sum_{n=m+1}^{\infty} c(m, m, m, n)t^n \tag{3.12}$$

In which the nature of $c(m, m, m, n)$ is not important to us.

Comparison of (3.11) and (3.12) leads to

$$s(m, m, m, n) = 0, \qquad for\ n < m \tag{3.13}$$

$$s(m, m, m, m) = a_0h_0^{3m}\gamma_m\delta_m\lambda_m \tag{3.14}$$

from which the conclusions in theorem 3 follow.

Theorem 4. For polynomial $f_n(x_1,x_2,x_3)$ defined by (3.1), with (3.2)-(3.9) holding and $\gamma_n \neq 0$, $\delta_n \neq 0$, $\lambda_n \neq 0$, there exist sequences of numbers α_k, β_k, ν_k, θ_k and μ_k such that, for $n \geq 1$

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}\right)f_n(x_1,x_2,x_3)-nf_n(x_1,x_2,x_3)$$

$$\begin{aligned}&=-\sum_{k=0}^{n-1}\alpha_k f_{n-1-k}(x_1,x_2,x_3)\\&\quad-\sum_{k=0}^{n-1}\beta_k\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}\right)f_{n-1-k}(x_1,x_2,x_3)\\&\quad-\sum_{k=0}^{n-1}\left(\nu_k\frac{\partial}{\partial x_1}+\theta_k\frac{\partial}{\partial x_2}+\mu_k\frac{\partial}{\partial x_3}\right)f_{n-1-k}(x_1,x_2,x_3)\end{aligned}\tag{3.15}$$

Indeed,

$$\frac{tA'(t)}{A(t)}=\sum_{n=0}^{\infty}\alpha_n t^{n+1}\tag{3.16}$$

$$\frac{tH'(t)}{H(t)}=1+\sum_{n=0}^{\infty}\beta_n t^{n+1}\tag{2.17}$$

$$\frac{tg'(t)}{H(t)}=\sum_{n=0}^{\infty}\nu_n t^{n+1}\tag{3.18}$$

$$\frac{tq'(t)}{H(t)}=\sum_{n=0}^{\infty}\theta_n t^{n+1}\tag{3.19}$$

$$\frac{tr'(t)}{H(t)}=\sum_{n=0}^{\infty}\mu_n t^{n+1}\tag{3.20}$$

Proof: Let

$$F = A(t)\phi_1[x_1H(t) + g(t)]\ \phi_2[x_2H(t) + q(t)]\phi_3[x_3H(t) + r(t)] \quad (3.21)$$

Then

$$\frac{\partial F}{\partial x_1} = H(t)A(t)\ \phi_1'\ \phi_2\ \phi_3 \quad (3.22)$$

$$\frac{\partial F}{\partial x_2} = H(t)A(t)\ \phi_1\ \phi_2'\ \phi_3 \quad (3.23)$$

$$\frac{\partial F}{\partial x_3} = H(t)A(t)\ \phi_1\ \phi_2\ \phi_3' \quad (3.24)$$

$$\frac{\partial F}{\partial t} = A'(t)\ \phi_1\ \phi_2\ \phi_3 + A(t)[x_1H'(t) + g'(t)]\ \phi_1'\ \phi_2\ \phi_3 + A(t)[x_2H'(t) + q'(t)]\ \phi_1\ \phi_2'\ \phi_3 + A(t)[x_3H'(t)+r'(t)]\ \phi_1\ \phi_2\ \phi_3' \quad (3.25)$$

Eliminating ϕ_1, ϕ_1', ϕ_2, ϕ_2', ϕ_3 and ϕ_3' from (3.21)-(3.25), we obtain

$$\left[\frac{x_1tH'(t)}{H(t)} + \frac{tg'(t)}{H(t)}\right]\frac{\partial F}{\partial x_1} + \left[\frac{x_2tH'(t)}{H(t)} + \frac{tq'(t)}{H(t)}\right]\frac{\partial F}{\partial x_2} + \left[\frac{x_3tH'(t)}{H(t)} + \frac{tr'(t)}{H(t)}\right]\frac{\partial F}{\partial x_3} - t\frac{\partial F}{\partial t} = -\frac{tA'(t)}{A(t)}F \quad (3.26)$$

Since

$$F = \sum_{n=0}^{\infty} f_n(x_1, x_2, x_3)t^n$$

It follows from (3.26) with the aid of (3.16)-(3.20) that

$$\left(1 + \sum_{n=0}^{\infty}\beta_n t^{n+1}\right)\left[\sum_{n=0}^{\infty}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right) f_n(x_1, x_2, x_3)t^n\right]$$

$$+\left(\sum_{n=0}^{\infty}\nu_n t^{n+1}\right)\left(\sum_{n=0}^{\infty}\frac{\partial}{\partial x_1}f_n(x_1,x_2,x_3)t^n\right)$$

$$+\left(\sum_{n=0}^{\infty}\theta_n t^{n+1}\right)\left(\sum_{n=0}^{\infty}\frac{\partial}{\partial x_2}f_n(x_1,x_2,x_3)t^n\right)$$

$$+\left(\sum_{n=0}^{\infty}\mu_n t^{n+1}\right)\left(\sum_{n=0}^{\infty}\frac{\partial}{\partial x_3}f_n(x_1,x_2,x_3)t^n\right)-\sum_{n=0}^{\infty}nf_n(x_1,x_2,x_3)t^n$$

$$=-\left(\sum_{n=0}^{\infty}\alpha_n t^{n+1}\right)\left(\sum_{n=0}^{\infty}f_n(x_1,x_2,x_3)t^n\right)$$

Therefore

$$\sum_{n=0}^{\infty}\left[\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}\right)f_n(x_1,x_2,x_3)-nf_n(x_1,x_2,x_3)\right]t^n$$

$$=-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\left\{(x_1\beta_k+\nu_k)\frac{\partial}{\partial x_1}+(x_2\beta_k+\theta_k)\frac{\partial}{\partial x_2}+(x_3\beta_k+\mu_k)\frac{\partial}{\partial x_3}\right\}\right.$$
$$\left. f_{n-k}(x_1,x_2,x_3)+\alpha_k f_{n-k}(x_1,x_2,x_3)\right]t^{n+1}$$

from which (3.15) follows after a shift from n to n-1 on the right.

The polynomials $g_n(x_1,x_2,x_3)$ of [9] fit into the above scheme with α_n=0, β_n=0, ν_0=-1, ν_n=0, θ_0=-1, θ_n=0, μ_0=-1, μ_n=0 for $n\geq 1$.

4. Generalization to m-Variable

Here we obtain m-variable analogues of theorems 1, 2, 3 and 4 mentioned above. Let $P_n(x_1,x_2,\cdots,x_m)$ be a polynomial in m-variables defined by means of the generating functions of the form

$$A(t)\ \prod_{j=1}^{m}\phi_j(x_jH(t))=\sum_{n=0}^{\infty}P_n(x_1,x_2,\cdots,x_m)t^n \tag{4.1}$$

and

$$\phi_j(t)=\sum_{n=0}^{\infty}\gamma_{j,n}\ t^n,\quad \gamma_{j,0}\ \neq\ 0\ ; j=1,2,\cdots,m \tag{4.2}$$

$$A(t)=\sum_{n=0}^{\infty}a_n\ t^n,\quad a_0\ \neq\ 0 \tag{4.3}$$

$$H(t)=\sum_{n=0}^{\infty}h_n\ t^{n+\frac{1}{m}},\ h_0\ \neq\ 0 \tag{4.4}$$

Then the following theorem holds:

Theorem 5. If $P_n(x_1,x_2,\cdots,x_m)$ in defined by (4.1), with (4.2), (4.3) and (4.4) holding, $P_n(x_1,x_2,\cdots,x_m)$ is a polynomial in x_1, x_2, $\cdots$, x_m and $P_n(x_1,x_2,\cdots,x_m)$ is of degree precisely n if and only if $\gamma_{j,n}\ \neq\ 0\ ;\ j=1,2,\cdots,m$.

Theorem 6. For polynomials $P_n(x_1,x_2,\cdots,x_m)$ defined by (4.1), with (4.2), (4.3) and (4.4) holding, and $\gamma_{j,n}\neq 0\ ;\ j=1,2,\cdots,m$, there exist sequences of numbers α_k and β_k such that, for $n\geq 1$.

$$\left[\sum_{j=1}^{m}x_j\frac{\partial}{\partial x_j}\right]P_n(x_1,x_2,\cdots,x_m)-nP_n(x_1,x_2,\cdots,x_m)$$

$$=-\sum_{k=0}^{n-1}\alpha_kP_{n-1-k}(x_1,x_2,\cdots,x_m)-\sum_{k=0}^{n-1}\beta_k\left[\sum_{j=1}^{m}x_j\frac{\partial}{\partial x_j}\right]P_{n-1-k}(x_1,x_2,\cdots,x_m) \tag{4.5}$$

Indeed,

$$\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^{n+1} \tag{4.6}$$

$$\frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \tag{4.7}$$

Example: Consider the polynomials $f_n(x_1, x_2, \cdots, x_m)$ of [13] in which

$$(1-t)^{-c} \prod_{j=1}^{m} \phi_j \left[\frac{-4x_j t}{(1-t)^2}\right] = \sum_{n=0}^{\infty} f_n(x_1, x_2, \cdots, x_m) t^n \tag{4.8}$$

The $f_n(x_1, x_2, \cdots, x_m)$ fit into the Boas & Buck theory with

$$A(t) = (1-t)^{-c}, \qquad H(t) = \frac{-4t}{(1-t)^2}$$

$$\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} c t^{n+1}, \quad \frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} 2t^{n+1}$$

Hence $\alpha_n = c$, $\beta_n = 2$ and the relation (4.5) becomes

$$\left[x_j \sum_{j=1}^{m} \frac{\partial}{\partial x_j}\right] P_n(x_1, x_2, \cdots, x_m) - nP_n(x_1, x_2, \cdots, x_m)$$

$$= -c \sum_{k=0}^{n-1} f_{n-1-k}(x_1, x_2, \cdots, x_m) - 2 \sum_{k=0}^{n-1} \left[\sum_{j=1}^{m} x_j \frac{\partial}{\partial x_j}\right] f_{n-1-k}(x_1, x_2, \cdots, x_m) \tag{4.9}$$

which is equation (3.6) of theorem of [13] with the right member written in reverse order.

The Boas and Buck type work obtained in the present paper for m-variable polynomials applies to polynomials considered in [11] and [13] but not to those of [9].

5. An Extension

Consider the generating relation

$$A(t)\prod_{j=1}^{m}\phi_j[x_jH(t)+g_j(t)]=\sum_{n=0}^{\infty}f_n(x_1,x_2,\cdots,x_m)t^n \tag{5.1}$$

In which

$$\phi_j(t)=\sum_{n=0}^{\infty}\gamma_{j,n}\ t^n,\qquad \gamma_{j,0}\ \neq\ 0\ ;\ j=1,2,\cdots,m \tag{5.2}$$

$$A(t)=\sum_{n=0}^{\infty}a_n\ t^n,\qquad a_0\ \neq\ 0 \tag{5.3}$$

$$H(t)=\sum_{n=0}^{\infty}h_n\ t^{n+\frac{1}{m}},\qquad h_0\ \neq\ 0 \tag{5.4}$$

$$g_j(t)=\sum_{n=0}^{\infty}g_{j,n}\ t^{n+2}\ ;\ j=1,2,\cdots,m \tag{5.5}$$

Note that $g_j(t)\ ;\ j=1,2,\cdots,m$ are permitted to be identically zero. It is not necessary to require that $g_j'(0)=0\ ;\ j=1,2,\cdots,m$ but these involve no loss of generality.

Theorem 7. If $f_n(x_1, x_2, \cdots, x_m)$ is defined by (5.1) with (5.2), (5.3), (5.4) and (5.5) holding, $f_n(x_1, x_2, \cdots, x_m)$ is a polynomial in $x_1,\ x_2,\ \cdots,$ x_m, and $f_n(x_1, x_2, \cdots, x_m)$ is of degree precisely n if and only if $\gamma_{j,n} \neq 0\ ;\ j = 1, 2, \cdots, m$.

Theorem 8. For polynomial $f_n(x_1, x_2, \cdots, x_m)$ defined by (5.1) with (5.2), (5.3), (5.4) and (5.5) holding and $\gamma_{j,n} \neq 0\ ;\ j = 1, 2, \cdots, m$, there exist sequences of numbers α_k, β_k and μ_k such that, for $n \geq 1$

$$\left[\sum_{j=1}^{m} x_j \frac{\partial}{\partial x_j}\right] f_n(x_1, x_2, \cdots, x_m) - n f_n(x_1, x_2, \cdots, x_m)$$

$$= -\sum_{k=0}^{n-1} \alpha_k f_{n-1-k}(x_1, x_2, \cdots, x_m)$$

$$-\sum_{k=0}^{n-1} \beta_k \left[\sum_{j=1}^{m} x_j \frac{\partial}{\partial x_j}\right] f_{n-1-k}(x_1, x_2, \cdots, x_m)$$

$$-\sum_{k=0}^{n-1} \left[\sum_{j=1}^{m} \mu_j \frac{\partial}{\partial x_j}\right] f_{n-1-k}(x_1, x_2, \cdots, x_m) \tag{5.6}$$

Indeed,

$$\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^n \tag{5.7}$$

$$\frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \tag{5.8}$$

$$\frac{tg'_j(t)}{H(t)} = \sum_{n=0}^{\infty} \mu_{j,n} t^{n+1} \ ; \ j = 1, 2, \cdots, m \qquad (5.9)$$

The polynomials $g_n(x_1, x_2, \cdots, x_m)$ of [9] fit into the above scheme with α_n=0, β_n=0, $\mu_{j,0} = -1$; and $\mu_{j,n} = 0$; $j = 1, 2, \cdots, m$ for $n \geq 1$.

References

[1] Boas, R.P. and Buck, R.C.: *Polynomials defined by generating functions*, Amer. Math. Monthly, Vol. 63 (1956), pp. 626-632.

[2] Chatterjea, S.K.: *A note on Laguerre polynomials of two variables*, Bull. Cal. Math. Soc., Vol. 83 (1991), pp. 263.

[3] Asenmyer, M.C. (Sister M. Celine): *Some generalized hypergeometric polynomials*, Bull. Amer. Math. Soc., Vol. 53 (1947), pp. 806-812.

[4] Khan, M.A. and Shukla, A.K.: *On Laguerre polynomials of several variables*, Bull. Cal. Math. Soc., Vol. 89 (1997), pp. 155-164.

[5] Khan, M.A. and Shukla, A.K.: *A note on Laguerre polynomials of m variables*, Bulletin of the Greek Mathematical Society., Vol. 40 (1998), pp. 113-117.

[6] Khan, M.A. and Abukhammash, G.S.: *On Hermite polynomials of two variables suggested by S.F. Ragab's Laguerre polynomials of two variables*, Bull. Cal. Math. Soc., Vol. 90 (1998), pp. 61-76.

[7] Khan, M.A. and Ahmad, K.: *On a general class of polynomials* $L_n^{(\alpha,\beta;\gamma,\delta)}(x,y)$ *of two variables suggested by the polynomials* $L_n^{(\alpha,\beta)}(x,y)$ *of Ragab and* $L_n^{(\alpha,\beta)}(x)$ *of Prabhakar and Rekha*, Pro Mathematica, Vol. XIX/Nos., 37-38 (2005), pp. 21-38.

[8] Khan, M.A. and Alidad, B.: *Polynomial sets generated by functions of the form* $G(2xt-t^2)K(2yt-t^2)$, Communicated for publication.

[9] Khan, M.A. and Alidad, B.: *Polynomial sets generated by functions of the form* $G_1(2x_1t-t^2)G_2(2x_2t-t^2)G_3(2x_3t-t^2)$ *and its generalization to m-variables*, Communicated for publication.

[10] Khan, M.A. and Alidad, B.: *Polynomial sets generated by* $e^t\phi(xt)\psi(yt)$, Communicated for publication.

[11] Khan, M.A. and Alidad, B.: *Polynomial sets generated by* $e^t\phi_1(x_1t)\phi_2(x_2t)\ \phi_3(x_3t)$ *and its generalization to m-variables*, Communicated for publication.

[12] Khan, M.A. and Alidad, B.: *Polynomial sets generated by functions of the form* $(1-t)^{-c}\ \phi\left[\frac{-4xt}{(1-t)^2}\right]\ \psi\left[\frac{-4yt}{(1-t)^2}\right]$, Communicated for publication.

[13] Khan, M.A. and Alidad, B.: *Polynomial sets generated by functions of the form* $(1-t)^{-c}\ \phi_1\left[\frac{-4x_1t}{(1-t)^2}\right]\ \psi_2\left[\frac{-4x_2t}{(1-t)^2}\right]\ \phi_3\left[\frac{-4x_3t}{(1-t)^2}\right]$ *and its generalization to m-variables*, Communicated for publication.

[14] Khan, M.A. and Alidad, B.: *Two variable analogue of Boas and Buck type Generating functions*, Communicated for publication.

[15] Ragab, S.F.: *On Laguerre polynomials of two variables* $L_n^{(\alpha,\beta)}(x,y)$, Bull. Cal. Math. Soc., Vol. 83 (1991), pp. 253.

[16] Rainville, E. D.: *Special Functions*, Macmillan, New York; Reprinted by Chelsea Publ. Co., Bronx., New York, (1971).

[17] Srivastava, H.M. and Manocha, H.L.: *A Treatise on Generating Functions*, John Wiley & Sons (Halsted Press), New York; Ellis Horwood, Chichester, 1984.

[18] Srivastava, H.M. and Karlsson, P.W.: *Multiple Gaussian hypergeometric series*, John Wiley & Sons (Halsted Press), New York; Ellis Horwood, Chichester, 1985.

Resumen

El presente artículo trata el análogo de tres variables de la función generatriz de Boas and Buck [14] para polinomios de dos variables y lo mismo se puede extender para el análogo de m variables. Los resultados obtenidos son extensiones de un artículo previo [14].

Palabras Clave: Funciones Generatrices del tipo Boas y Buck, conjuntos de polinomios de tres variables, conjuntos de polinomios de m variables.

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