ON RARELY $\beta\theta$-CONTINUOUS FUNCTIONS

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Abstract

Popa [13] introduced the notion of rare continuity. In the same spirit, we introduce a new class of functions called rarely $\beta\theta$-continuous functions by utilizing the notion of β-θ-open sets. We also investigate some of its fundamental properties. This type of continuity is stronger that rare β-continuity [9].

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1. Introduction and Preliminaries

Popa [13] introduced the notion of rarely continuity as a generalization of weak continuity [10] which has been further investigated by Long and Herrington [11] and Jafari [7] and [8]. Jafari [9] also generalized the notion of rare continuity to rare β-continuity by involving the notion of β-open sets. The purpose of the present paper is to introduce the concept of rare $\beta\theta$-continuity in topological spaces as a notion stronger of rare β-continuity by utilizing the notion of β-θ-open sets introduced by Noiri [12] and investigated also by Caldas [3, 5, 4, 6]. We investigate several properties of rarely $\beta\theta$-continuous functions. The notion of $I.\beta\theta$-continuity is also introduced which is stronger than rare $\beta\theta$-continuity.

Throughout this paper, (X, τ) and (Y, σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Abd El Monsef et al. [1] and Andrijević [2] introduced the notion of β-open set, which Andrijević called semipreopen, completely independent of each other. In this paper, we adopt the word β-open for the sake of clarity. A subset A of a topological space (X, τ) is called β-open if $A \subseteq Cl(Int(Cl(A)))$, where $Cl(A)$ and $Int(A)$ denote the closure and the interior of A, respectively. The complement of a β-open set is called β-closed. The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by $\beta Cl(A)$. The family of all open and β-open sets will be denoted by $O(X)$ and $\beta O(X)$ respectively. We set $O(X, x) = \{U \mid x \in U \in O(X)\}$ and $\beta O(X, x) = \{U \mid x \in U \in \beta O(X)\}$. Recall that a rare set is a set R such that $Int(R) = \emptyset$.

Now we begin to recall some known notions which will be used in the sequel.

Definition 1. *[12]. Let A a subset of X. The β-θ-*closure *of A, denoted by $\beta Cl_\theta(A)$, is the set of all $x \in X$ such that $\beta Cl(O) \cap A \neq \emptyset$ for every $O \in \beta O(X, x)$. A subset A is called β-θ-*closed *if $A = \beta Cl_\theta(A)$. The set*

$\{x \in X \mid \beta Cl_\theta(O) \subset A$ *for some* $O \in \beta O(X, x)\}$ *is called the* β θ-interior *of* A *and is denoted by* $\beta Int_\theta(A)$. *A subset* A *is called* β-θ-open *if* $A = \beta Int_\theta(A)$. *The family of all* β-θ-*open sets will be denoted by* $\beta\theta O(X)$. *We set* $\beta\theta O(X, x) = \{U \mid x \in U \in \beta\theta O(X)\}$.

The following theorem is know and given by Noiri [12].

Theorem 1.1. *For any subset* A *of* X*:*

(1) $\beta Cl_\theta(\beta Cl_\theta(A)) = \beta Cl_\theta(A)$.

(2) $\beta Cl_\theta(A)$ *is* β-θ-*closed.*

(3) Intersection of arbitrary collection of β-θ-*closed set in* X *is* β-θ-*closed.*

(4) $\beta Cl_\theta(A)$ *is the intersection of all* β-θ-*closed sets each containing* A.

(5) If $A \in \beta O(X)$ *then,* $\beta Cl(A) = \beta Cl_\theta(A)$.

Definition 2. *A function* $f : X \to Y$ *is called:*

i) Weakly continuous [10] (resp. weakly-$\beta\theta$*-continuous [12]) if for each* $x \in X$ *and each open set* G *containing* $f(x)$*, there exists* $U \in O(X, x)$ *(resp.* $U \in \beta\theta O(X, x)$*) such that* $f(U) \subset Cl(G)$.

ii) Rarely continuous [13](resp. rarely β*-continous [9]) if for each* $x \in X$ *and each* $G \in O(Y, f(x))$*, there exist a rare set* R_G *with* $G \cap Cl(R_G) = \emptyset$ *and* $U \in O(X, x)$ *(resp.* $U \in \beta O(X, x)$ *such that* $f(U) \subset G \cup R_G$.

2. Rare $\beta\theta$-continuity

Definition 3. *A function* $f : X \to Y$ *is called rarely* $\beta\theta$*-continuous if for each* $x \in X$ *and each* $G \in O(Y, f(x))$*, there exist a rare set* R_G *with* $G \cap Cl(R_G) = \emptyset$ *and* $U \in \beta\theta O(X, x)$ *such that* $f(U) \subset G \cup R_G$.

By V. Popa and T. Noiri [[14], Remark 2.6], S. Jafari [9] and Definition 2, we obtained the following diagram:

$$\begin{array}{ccccc}
\text{continuity} & \rightarrow & \text{weak continuity} & \rightarrow & \text{rare continuity} \\
 & & & & \downarrow \\
 & & \text{rare } \beta\theta\text{-continuity} & \rightarrow & \text{rare } \beta\text{-continuity}
\end{array}$$

Example 2.1. *(I) Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. We obtain:*
$\beta O(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\} = \beta\theta O(X, \tau)$.
Then, f is not continuous and not rarely $\beta\theta$-Continuous, but it is weakly continuous, rarely continuous and rarely β Continuous.

(II) Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. We obtain:
$\beta O(X, \tau) = \beta\theta O(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$
Then, f is not rarely continuous but, f is rarely $\beta\theta$-Continuous.

Theorem 2.2. *The following statements are equivalent for a function $f : X \rightarrow Y$:*

(1) f is rarely $\beta\theta$-continuous at $x \in X$.

(2) For each set $G \in O(Y, f(x))$, there exists $U \in \beta\theta O(X, x)$ such that $Int[f(U) \cap (Y \setminus G)] = \emptyset$.

(3) For each set $G \in O(Y, f(x))$, there exists $U \in \beta\theta O(X, x)$ such that $Int[f(U)] \subset Cl(G)$.

(4) For each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ such that $x \in \beta Int_\theta(f^{-1}(G \cup R_G))$.

(5) For each $G \in O(Y, f(x))$, there exists a rare set R_G with $Cl(G) \cap R_G = \emptyset$ such that $x \in \beta Int_\theta(f^{-1}(Cl(G) \cup R_G))$.

(6) For each $G \in RO(Y, f(x))$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ such that $x \in \beta Int_\theta(f^{-1}(G \cup R_G))$.

Proof.

$(1) \Rightarrow (2):$ Let $G \in O(Y, f(x))$. By $f(x) \in G \subset Int(Cl(G))$ and the fact that $Int(Cl(G)) \in O(Y, f(x))$, there exist a rare set R_G with $Int(Cl(G)) \cap Cl(R_G) = \emptyset$ and a β-θ-open set $U \subset X$ containing x such that $f(U) \subset Int(Cl(G)) \cup R_G$. We have $Int[f(U) \cap (Y - G)] = Int[f(U)] \cap Int(Y - G) \subset Int[Cl(G) \cup R_G] \cap (Y - Cl(G)) \subset (Cl(G) \cup Int(R_G)) \cap (Y - Cl(G)) = \emptyset$.

$(2) \Rightarrow (3):$ It is straightforward.

$(3) \Rightarrow (1):$ Let $G \in O(Y, f(x))$. Then by (3), there exists $U \in \beta\theta O(X, x)$ such that $Int[f(U)] \subset Cl(G)$. We have $f(U) = [f(U) - Int(f(U))] \cup Int(f(U)) \subset [f(U) - Int(f(U))] \cup Cl(G) = [f(U) - Int(f(U))] \cup G \cup (Cl(G) - G) = [f(U) - Int(f(U))] \cap (Y - G) \cup G \cup (Cl(G) - G)$. Set $R^* = [f(U) - Int(f(U))] \cap (Y - G)$ and $R^{**} = (Cl(G) - G)$. Then R^* and R^{**} are rare sets. Moreover $R_G = R^* \cup R^{**}$ is a rare set such that $Cl(R_G) \cap G = \emptyset$ and $f(U) \subset G \cup R_G$. This shows that f is rarely- $\beta\theta$-continuous.

$(1) \Rightarrow (4):$ Suppose that $G \in O(Y, f(x))$. Then there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $U \in \beta\theta O(X, x)$ such that $f(U) \subset G \cup R_G$. It follows that $x \in U \subset f^{-1}(G \cup R_G)$. This implies that $x \in \beta Int_(\theta)(f^{-1}(G \cup R_G))$.

$(4) \Rightarrow (5):$ Suppose that $G \in O(Y, f(x))$. Then there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ such that $x \in \beta Int_\theta(f^{-1}(G \cup R_G))$. Since $G \cap Cl(R_G) = \emptyset$, $R_G \subset Y - G$, where $Y - G = (Y - Cl(G)) \cup (Cl(G) - G)$. Now, we have $R_G \subset (R_G \cup (Y - Cl(G)) \cup (Cl(G) - G))$. Set $R^* = R_G \cap (Y - Cl(G))$. It follows that R^* is a rare set with $Cl(G) \cap R^* = \emptyset$. Therefore $x \in \beta Int_\theta[f^{-1}(G \cup R_G)] \subset \beta Int_\theta[f^{-1}(Cl(G) \cup R^*)]$.

(5) $\Rightarrow$ (6) : Assume that $G \in RO(Y, f(x))$. Then there exists a rare set R_G with $Cl(G) \cap R_G = \emptyset$ such that $x \in \beta Int_\theta[f^{-1}(Cl(G) \cup R_G)]$. Set $R^* = R_G \cup (Cl(G) - G)$. It follows that R^* is a rare set and $G \cap Cl(R^*) = \emptyset$. Hence
$x \in \beta Int_\theta[f^{-1}(Cl(G) \cup R_G)] = \beta Int_\theta[f^{-1}(G \cup (Cl(G) - G) \cup R_G)] = \beta Int_\theta[f^{-1}(G \cup R^*)]$.

(6) $\Rightarrow$ (1) : Let $G \in O(Y, f(x))$. By $f(x) \in G \subset Int(Cl(G))$ and the fact that $Int(Cl(G)) \in RO(Y)$, there exists a rare set R_G and $Int(Cl(G)) \cap Cl(R_G) = \emptyset$ such that $x \in \beta Int_\theta[f^{-1}(Int(Cl(G)) \cup R_G)]$. Let $U = \beta Int_\theta[f^{-1}(Int(Cl(G)) \cup R_G]$ and $U \in \beta\theta O(X, x)$. Therefore $f(U) \subset Int(Cl(G)) \cup R_G$. Hence, we have $Int[f(U) \cap (Y - G)] = \emptyset$ and therefore, f is rarely $\beta\theta$-continuous.

Definition 4. *A function $f : X \to Y$ is $I.\beta\theta$-continuous at $x \in X$ if for each set $G \in O(Y, f(x))$, there exists $U \in \beta\theta O(X, x)$ such that $Int[f(U)] \subset G$.*
If f has this property at each point $x \in X$, then we say that f is $I.\beta\theta$-continuous on X.

It should be noted that $I.\beta\theta$-continuity is stronger than rare $\beta\theta$-continuity.

Question. Are there examples showing that a function is rarely $\beta\theta$-continuous but not $I.\beta\theta$-continuous?

Theorem 2.3. *Let Y be a regular space. Then the function $f : X \to Y$ is $I.\beta\theta$-continuous on X if and only if f is rarely $\beta\theta$-continuous on X.*

Proof. We prove only the sufficient condition since the necessity condition is evident.

Let f be rarely $\beta\theta$-continuous on X and $x \in X$. Suppose that $f(x) \in G$, where G is an open set in Y. By the regularity of Y, there exists an open set $G_1 \in O(Y, f(x))$ such that $Cl(G_1) \subset G$. Since f is rarely $\beta\theta$-continuous, then there exists $U \in \beta\theta O(X, x)$ such that

$Int[f(U)] \subset Cl(G_1)$ (Theorem 2.2). This implies that $Int[f(U)] \subset G$ and therefore f is $I.\beta\theta$-continuous on X.

We say that a function $f : X \to Y$ is $r.\beta\theta$-open if the image of a β-θ-open set is open.

Theorem 2.4. *If $f : X \to Y$ be an $r.\beta\theta$-open rarely $\beta\theta$-continuous function, then f is weakly $\beta\theta$-continuous.*

Proof. Suppose that $x \in X$ and $G \in O(Y, f(x))$. Since f is rarely $\beta\theta$-continuous, there exists a rare set R_G with $Cl(R_G) \cap U = \emptyset$ where $U \in \beta\theta O(X, x)$ such that $f(U) \subset G \cup R_G$. This means that $(f(U) \cap (Y \setminus Cl(G)) \subset R_G$. Since the function f is $r.\beta\theta$-open, then $f(U) \cap (Y \setminus Cl(G))$ is open. But the rare set R_G has no interior points. Then $f(U) \cap (Y \setminus Cl(G)) = \emptyset$. This implies that $f(U) \subset Cl(G)$ and thus f is weakly $\beta\theta$-continuous.

Definition 5. *Let $A = \{G_i\}$ be a class of subsets of X. By rarely union sets [7] of A we mean $\{G_i \cup R_{G_i}\}$, where each R_{G_i} is a rare set such that each of $\{G_i \cap Cl(R_{G_i})\}$ is empty.*

Recall that, a subset B of X is said to be rarely almost compact relative to X [7] if every open cover of B by open sets of X, there exists a finite subfamily whose rarely union sets cover B.
A topological space X is said to be rarely almost compact [7] if the set X is rarely almost compact relative to X.
A topological space X is called $\beta\theta O$-compact if every cover of X by β-θ-open sets has a finite subcover.

Theorem 2.5. *Let $f : X \to Y$ be rarely $\beta\theta$-continuous and K an $\beta\theta O$-compact set relative to X. Then $f(K)$ is rarely almost compact subset relative to Y.*

Proof. Suppose that Ω is a open cover of $f(K)$. Let B be the set of all V in Ω such that $V \cap f(K) \neq \emptyset$. Then B is an open cover of $f(K)$. Hence for each $k \in K$, there is some $V_k \in B$ such that $f(k) \in V_k$. Since f is rarely $\beta\theta$-continuous there exist a rare set R_{V_k} with $V_k \cap Cl(R_{V_k}) = \emptyset$ and a β-θ-open set U_k containing k such that $f(U_k) \subset V_k \cup R_{V_k}$. Hence there is a finite subfamily $\{U_k\}_{k\in\triangle}$ which covers K, where $\triangle$ is a finite subset of K. The subfamily $\{V_k \cup R_{V_k}\}_{k\in\triangle}$ also covers $f(K)$.

Recall that a space X is called βT_θ-space [3] if every β-θ-closed set in X is closed in X.

Theorem 2.6. *Let $f : X \to Y$ be rarely $\beta\theta$-continuous and X a βT_θ-space. Then f is rarely continuous.*

Lemma 2.7. *(Long and Herrington [11]). If $g : Y \to Z$ is continuous and one-to-one, then g preserves rare sets.*

Theorem 2.8. *If $f : X \to Y$ is rarely $\beta\theta$-continuous and $g : Y \to Z$ is continuous and one-to-one, the $g \circ f : X \to Z$ is rarely $\beta\theta$-continuous.*

Proof. Suppose that $x \in X$ and $(g \circ f)(x) \in V$, where V is an open set in Z. By hypothesis, g is continuous, therefore there exists an open set $G \subset Y$ containing $f(x)$ such that $g(G) \subset V$. Since f is rarely $\beta\theta$-continuous, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and an β-θ-open set U containing x such that $f(U) \subset G \cup R_G$. It follows from Lemma 2.7 that $g(R_G)$ is a rare set in Z. Since R_G is a subset of $Y \setminus G$ and g is injective, we have $Cl(g(R_G)) \cap V = \emptyset$. This implies that $(g \circ f)(U) \subset V \cup g(R_G)$. Hence the result.

Recall, that a function $f : X \to Y$ is called pre-$\beta\theta$-open [4] if $f(U)$ is β-θ-open in Y for every β-θ-open set U of X.

Theorem 2.9. *Let $f : X \to Y$ be pre-$\beta\theta$-open and $g : Y \to Z$ a function such that $g \circ f : X \to Z$ is rarely $\beta\theta$-continuous. Then g is rarely $\beta\theta$-continuous.*

Proof. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let $G \in O(Z, (g \circ f)(x))$. Since $g \circ f$ is rarely $\beta\theta$-continuous, there exists a rare set R_G

with $G \cap Cl(R_G) = \emptyset$ and $U \in \beta\theta O(X, x)$ such that $(g \circ f)(U) \subset G \cup R_G$. But $f(U)$ (say V) is a β-θ-open set containing $f(x)$. Therefore, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $V \in \beta\theta O(Y, y)$ such that $g(V) \subset G \cup R_G$, i.e., g is rarely $\beta\theta$-continuous.

Definition 6. *A space X is called r-separate [8] if for every pair of distinct points x and y in X, there exist rare sets R_{U_x}, R_{U_y} and open sets U_x and U_y with $U_x \cap Cl(R_{U_x}) = \emptyset$ and $U_y \cap Cl(R_{U_y}) = \emptyset$ such that $(U_x \cup R_{U_x}) \cap (U_y \cup R_{U_y}) = \emptyset$.*

In [13], Popa obtained the following result.

Theorem 2.10. *The function $f : X \to Y$ is rarely continuous if and only if for each open set $G \subset Y$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ such that $f^{-1}(G) \subset Int[f^{-1}(G \cup R_G)]$.*

Theorem 2.11. *If Y is r-separate and $f : X \to Y$ is rarely $\beta\theta$-continuous injection and X a βT_θ-space, then X is Hausdorff.*

Proof. Since f is injective, then $f(x) \neq f(y)$ for any distinct points x and y in X. Since Y is r-separate, There exist open sets G_1 and G_2 in Y containing $f(x)$ and $f(y)$, respectively, and rare sets $R_{G_1}$and R_{G_2} with $G_1 \cap Cl(R_{G_1}) = \emptyset$ and $G_2 \cap Cl(R_{G_2}) = \emptyset$ such that $(G_1 \cup R_{G_1}) \cap (G_2 \cup R_{G_2}) = \emptyset$. Therefore $Int[f^{-1}(G_1 \cup R_{G_1})] \cap Int[f^{-1}(G_2 \cup R_{G_2})] = \emptyset$. Since X is βT_θ-space, then every rarely $\beta\theta$-continuous is rarely continuous and by Theorem 2.11, we have $x \in f^{-1}(G_1) \subset Int[f^{-1}(G_1 \cup R_{G_1})]$ and $y \in f^{-1}(G_2) \subset Int[f^{-1}(G_2 \cup R_{G_2})]$, This shows that X is Hausdorff.

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Resumen

En este paper nosotros introducimos una nueva clase de función denominada función raramente $\beta\theta$-continua usando la noción de conjuntos β-θ-abiertos. Este tipo de continuidad es una noción más fuerte que la noción de función rare β-continua dada por Jafari [9].

Palabras clave: Conjuntos raros, conjuntos β-θ-abiertos, funciones raramente continuas.

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