THE METHOD OF MIXED MONOTONY AND FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract

In this paper I extend the method of mixed monotony, to construct monotone sequences that converge to the unique solution of an initial value delay differential equation.

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1. Introduction

One of the most useful methods in proving existence of solutions of delay differential equation, initial and boundary value problems is the monotone iterative technique. The method of upper and lower solutions coupled with the monotone iterative technique offers a flexible and effective mechanism for proving constructive existence result in a sector. The upper and lower solutions that generate the sector serve as upper and lower bounds for solutions which can be improved by the monotone iterative procedure. Moreover, the iterative schemes are also useful to investigate qualitative properties of solutions. In recent years, the ideas imbedded in these combined techniques have played an important role in unifying a variety of nonlinear problems and have also proved to be of immense value [1].

Consider the initial delay differential equation

$$u' = f(t, u(t), u_t), u_t = \varphi \in C,$$

where

$$f \in C[I_0 \times R \times C, R], C = C([-\tau, 0], R), I_0 = [t_0, t_0 + T], \text{ for } t_0 \geq 0 (1.1)$$

In order to develop monotone iterative technique for (1.1) f should be at least mixed quasi-monotone in which case employing the notion of quasi-solutions and the theory of mixed monotone operators, one can construct monotone sequences that converge to quasi-solutions (1.1), if further f satisfies a uniqueness condition, then it can be shown that monotone sequences converge to the unique solution of (1.1) in the sector [1].

The question of whether it is possible to construct monotone sequences that converge to the unique solutions of (1.1) even when f does not possess any monotone properties is natural and interesting. See [4]. See [5] - [13] for related results and notation.

In papers [1], [2], and [3], the authors have developed a new method, called the method of mixed monotony, which makes it possible to construct monotone sequences that converge to the unique solution of initial and boundary value problems. In this paper, I extend this method to an initial value problem for delay differential equations.

2. Method of Mixed Monotony for DDES

Given the delay differential equation (DDE)

$$u' = f(t, u(t), u_t), u_t = \varphi \in C \tag{2.1}$$

where $f \in C[I_0 \times R \times C, R], C = C([-\tau, 0], R), I_0 = [t_0, t_0 + T]$, for $t_0 \geq 0$. and for any $t \in I_0, u_t \in C$ is defined by $u_t(s) = u(t+s), -\tau \leq s \leq 0$.

Suppose that there exist functions

$$F \in C[I \times R \times R \times C, R] \text{ and } \alpha, \beta \in C[I, R] \cap C'[I_0, R],$$

satisfying the following conditions:

(I) $\alpha' \leq F(t, \alpha, \beta, \alpha_t), \beta' \geq F(t, \beta, \alpha, \beta_t)$

(II) F is mixed monotone so $F(t, x, y, z)$ is monotone nondecreasing in x, and monotone nondecreasing in y.

(III) F is locally Lipschitzian on I relative to the pair (α, β).

(IV) If $\ell, k \in [I, R]$ such that $\ell \leq k$ on I $\ell' = F(t, \ell, k, \ell_t), \ell_{t_0} = \phi$, and $k' = F(t, r, \ell, r_t), r_{t_0} = \phi$, then $\ell = k$ on I.

Then we say that the DDE (2.1) admits a *process of mixed monopolization.*

To prove the main result, we need the following comparison result; see [7].

Lemma 2.1. *Assume that*

$$p' \leq -Mp(t) - N \int_{-\tau}^{0} p_t(s)ds, \;\; \textit{on } I_0 = t[t_0, t_0 + T] \tag{2.2}$$

Suppose further that

(I) Either $p(t_0) \leq p_{t_0}(s) \leq 0, s \in [-\tau, 0]$ and

$$(M + N\tau)T \leq 1 \text{ or} \tag{2.3}$$

(II) $P_{t_0}(s) \leq 0, s \in [-\tau, 0], p \in c'[t_0 - \tau, t_0]$ and

$$p'(t) \leq \frac{\lambda}{T + \tau}, \;\; t \in [t_0 - \tau, t_0] \tag{2.4}$$

where $\min\limits_{[t_0 - \tau, t_0]} p(t) = -\lambda, \;\; \lambda \geq 0$

$$(M + N\tau)(T + \tau) \leq 1. \tag{2.5}$$

Then $p(t) \leq 0$ on I_0.

Proof.
Assume that the conclusion of the lemma is false. Let us consider the case (I). Then there exist $t_1, t_2 \in I_0$ such that $t_1 < t_2, \; p(t_2) > 0$ and

$$\min_{[t_0 - \tau, t_2]} p(t) = -\lambda = p(t_1) \leq 0$$

Let us first consider the case $\lambda > 0$. By the mean value theorem, there exists a $\bar{t} \in [t_1, t_2]$ such that

$$p'(\bar{t}) = \frac{p(t_2) - p(t_1)}{t_2 - t_1} > \frac{\lambda}{T}.$$

On the other hand, we have

$$p'(\bar{t}) \leq -Mp(\bar{t}) - N \int_{-\tau}^{0} p_{\bar{t}}(s)ds \leq (M + N\tau)\lambda$$

which contradicts the assumption (2.3). If $\lambda = 0$, a contradiction is obtained also. In case of (II) minimal value of $p(t)$ on $[t_0 - \tau, t_2]$ lies on $[t_0 - \tau, t_0]$. Hence using mean value theorem on $[t', t_2]$, where $-\lambda = p(t')t' \in [t_0 - \tau, t_0]$ and $p(t')$ is minimal value of $p(t)$ on $[t_0 - \tau, t_0]$, there exist a $\bar{\bar{t}} \in [t', t_2]$, such that

$$p'\left(\bar{\bar{t}}\right) = \frac{p(t_2) - p(t')}{t_2 - t'} > \frac{\lambda}{T + \tau} \tag{2.6}$$

There are two possibilities, namely $\bar{\bar{t}} \in [t_0, t_2]$ and $\bar{\bar{t}} \in [t_0 - \tau, t_0]$. If $\bar{\bar{t}} \in [t_0, t_2]$ then by using (2.2) we get

$$p'\left(\bar{\bar{t}}\right) \leq (M + N\tau)\lambda$$

which in view of (2.6) contradicts (2.5). If

$$\bar{\bar{t}} \in [t_0 - \tau, t_0],$$

then (2.6) contradicts (2.4). The Proof is therefore complete.

Theorem 2.1. *Consider the DDE (2.1) and assume that DDE (2.1) admits a process of mixed monopolization. Then there exist monotone sequences $\{\alpha_n(t), \beta_n(t)\}$ which converge uniformly on I_0 to the unique solution $u(t)$ of DDE (2.1).*

Furthermore, we have

$$\alpha(t) \leq \alpha_1(t) \leq \ldots \leq \alpha_n(t) \leq \ldots \leq \beta_1(t) \leq \beta(t) \,\text{on}\, I_0$$

Proof.
Let $\eta, \mu \in C[I, R], \eta, \mu \in [\alpha, \beta] = \{u \in C[I, R] : \alpha(t) \leq u(t) \leq \beta(t) \,\text{on}\, I\}$. Consider the linear DDE:

$$\begin{aligned} \bar{u}' &= F(t, \eta, \mu, \eta) + M(\eta - \bar{u}) + N \int_{-\tau}^{0} [\eta_t(s) - \bar{u}_t(s)]ds, \\ \bar{u}_{t_0} &= \phi_0. \end{aligned} \tag{2.7}$$

Given any η, μ, using standard method of steps clearly there exists a unique solution u of the linear DDE (2.7) on I.

Pick $\eta = \alpha, \mu = \beta$, and $\bar{u'} = \alpha_1'$, then equation (2.7) becomes

$$\begin{aligned}\alpha_1' &= F(t,\alpha,\beta,\alpha) + M(\alpha-\alpha_1) + N\int_{-\tau}^{0}[\alpha_t(s)-\alpha_{1t}(s)]ds,\\ \alpha_{1t_0} &= \phi_0.\end{aligned} \tag{2.8}$$

Equation (2.8) and assumption (I) imply that:

$$\alpha' - \alpha_1' \leq -M(\alpha-\alpha_1) - N\int_{-\tau}^{0}[\alpha_t(s)-\alpha_{1t}(s)]ds,$$

which reduces to

$$p' \leq -Mp - N\int_{-\tau}^{0} p_t(s)ds,\ \ p_{t_0} = \phi_0 - \alpha_{t_0} \leq 0. \tag{2.9}$$

By Lemma (2.1), we have $p(t) = \alpha - \alpha_1 \leq 0$, which implies that $\alpha \leq \alpha_1$.

Choose $\eta = \beta, \mu = \beta$, and $\bar{u'} = \beta'$, then equation (2.2) yields

$$\beta_1' = F(t,\beta,\alpha,\beta) + M(\beta-\beta_1) + N\int_{-\tau}^{0}[\beta_t(s)-\beta_{1t}(s)]ds, \tag{2.10}$$

Equation (2.10) and (I) give

$$\beta_1' - \beta \leq -M(\beta_1-\beta) - N\int_{-\tau}^{0}[\beta_{1t}(s)-\beta_t(s)]ds,$$

which yields $\beta_1 \leq \beta$.

Let $\eta_1 \leq \eta_2$, $\eta_1, \eta_2 \in C[I, R]$ and $\eta_1, \eta_2 \in [\alpha, \beta]$.

Consider

$$v' = F(t,\eta_1,\mu_1,\eta_2) + M(\eta_1 - v) + N\int_{-\tau}^{0}[\eta_{1t}(s)-v_t(s)]ds, \tag{2.11}$$

$$w' = F(t,\eta_2,\mu_1,\eta_1) + M(\eta_2 - w) + N\int_{-\tau}^{0}[\eta_{2t}(s)-w_t(s)]ds, \tag{2.12}$$

$$R = v - w. \tag{2.13}$$

By using mixed monotony of F, equation (2.10), (2.11), (2.12), and Lemma (2.1), we get $v \leq w$ for all $t_0 \leq t \leq t_1$.

Similarly if we switch the roles of η and μ and use mixed monotony of F, one can prove that $v \geq w$.

By defining a mapping $A[\eta, \mu] = u$, where u is the unique solution to equation

$$\begin{aligned} u' &= F(t, \eta, \mu, \eta) + M(\eta - u) + N\int_{-\tau}^{0} [\eta_t(s) - u_t(s)]ds, \qquad (2.14) \\ u_{t_0} &= \phi_0. \end{aligned}$$

and by taking into account that $\alpha \leq \alpha_1, \beta \geq \beta_1, v \leq w$, where $\eta_1 \leq \eta_2$, we can conclude that

(a) $\alpha \leq A[\alpha, \beta]$

(b) $\beta \geq A[\beta, \alpha]$

(c) $A[\eta_1, \mu] \leq A[\eta_2, \mu]$ if $\eta_1 \leq \eta_2$

(d) $A[\eta, \mu_1] \leq A[\eta, \mu_2]$ if $\mu_1 \geq \mu_2$.

Define the sequences $\{\alpha_n\}, \{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, by $\alpha_{n+1} = A[\alpha_n, \beta_n]$ and, $\beta_{n+1} = A[\beta_n, \alpha_n]$.

Using (a), (b) and, (c), we can conclude that $\alpha \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_1 \leq \beta$ on I. By a standard argument we can show $\lim_{n\to\infty} \alpha_n = \ell$, $\lim_{n\to\infty} \beta_n = k$, exist uniformly on I, and (ℓ, k) satisfy $\ell' = F(t, \ell, k, \ell)$, $k' = F(t, k, \ell, k)$. From (IV) we see that $u = \ell = k$ is the unique solution of DDE (2.1). The proof is complete.

Theorem 2.2. *Consider the DDE (2.1).*

Suppose that

(H_1) $\alpha, \beta \in C[I, R];\ \alpha \leq \beta(t)$ on I

$$\begin{aligned} \alpha' &\leq f(t, \alpha, \alpha_t) - M(\beta - \alpha) - N \int_{-\tau}^{0} (\beta_t(s) - \alpha_t(s))ds \\ \beta' &\geq f(t, \beta, \beta_t) + M(\beta - \alpha) + N \int_{-\tau}^{0} (\beta_t(s) - \alpha_t(s))ds \end{aligned}$$

(H_2) f satisfy $|f(t, x, \phi_2) - f(t, y, \phi_1)| \leq M(x - y) + N \int_{-\tau}^{0} (\phi_2(s) - \phi_1(s))ds$ whenever $\alpha(t) \leq y \leq x \leq \beta(t), \alpha_t \leq \phi_1 \leq \phi_2 \leq \beta_t$ on I_0 and $M, N \in R^+$.

Then DDE (2.1) admits the method of mixed monotony.

Proof.
Define

$$F(t, y, z, y_t) = \frac{1}{2}[f(t, y, y_t) + f(t, z, z_t)] + M(y - z) + N \int_{-\tau}^{0} [y_t(s) - z_t(s)]ds.$$

It is easy to see that F is mixed monotone and locally Lipschitzian.

$$F(t, y, z, y_t) - F(t, \bar{y}, \bar{z}, \bar{y}_t) \geq M(y - \bar{y}) + N \int_{-\tau}^{0} [y_t(s) - \bar{y}_t(s)]ds, \quad (2.15)$$

whenever $\alpha(t) \leq \bar{y} \leq y \leq \beta(t)$, and $\alpha_t(t) \leq \bar{y}_t \leq y_t \leq \beta_t(t)$

In particular we have

$$F(t, y, z, y_t) - F(t, z, y, z_t) = M(y - z) + N \int_{-\tau}^{0} (y_t(s) - z_t(s))ds \quad (2.16)$$

From (2.13),

$$F(t, \alpha, \beta, \alpha_t) - F(t, \beta, \alpha, \beta_t) \geq -M(\alpha - \beta) - N \int_{-\tau}^{0} (\alpha_t(s) - \beta_t(s))ds, \text{ or}$$

$$F(t, \alpha, \beta, \alpha_t) + M(\beta - \alpha) - N \int_{-\tau}^{0} (\beta_t(s) - \alpha_t(s))ds \leq F(t, \alpha, \beta, \alpha_t),$$

this yields $\alpha' \leq F(t, \alpha, \beta, \alpha_t)$. Similarly we get $\beta' \geq F(t, \beta, \alpha, \beta_t)$.

Finally $F(t, x, x, x_t) = F(t, x, x_t)$ by setting $x = y = z$, and the proof is complete.

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Resumen

En este artículo se prueba una generalización del método de monotonía mixta, para construir sucesiones monótonas que convergen a la solución única de una ecuación diferencial de retraso con valor inicial.

Palabras clave: Ecuación diferencial de retraso, operador de monotonía mixto, procesos iterativos

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