ON A GENERALIZATIONS OF LAURICELLA'S FUNCTIONS OF SEVERAL VARIABLES

Mumtaz Ahmad Khan [1] ***and K. S. Nisar*** [1]

October, 2010

Abstract

The present paper introduces 10 Appell's type generalized functions N_i, $i = 1, 2, \ldots\ldots 10$ by considering the product of $n - {}_3F_2$ functions. The paper contains Fractional derivative representations, Integral representations and symbolic forms similar to those obtained by J. L. Burchnall and T. W. Chaundy for the four Appell's functions, have been obtained for these newly defined functions $N_1, N_2 \ldots\ldots\ldots N_{10}$. The results obtained are believed to be new.

MSC(2010): 42C05, 33C45.

Keywords: *Hypergeometric Series, Lauricella's functions, Fractional derivatives and Integral representations.*

1. *Department of Applied Mathematics, Aligarh Muslim Univ. India.*

1. Introduction

The great success of the theory of hypergeometric functions of a single variable has stimulated the development of a corresponding theory in two or more variables. In 1880, P. Appell [1] considered the product of two Gauss functions from which the four Appell's functions emerged. Later 1893, Lauricella [7] further generalized the four Appell functions F_1, F_2, F_3 and F_4 to function of n-variable denoted by F_A^n, F_B^n, F_C^n and F_D^n, where $F_A^1 = F_B^1 = F_C^1 = F_D^1 = {}_2F_1$ and $F_A^2 = F_2$, $F_B^2 = F_3$, $F_C^2 = F_3$ and $F_D^2 = F_1$.

During 1940-41, J. L. Burchnall and T. W. Chaundy [2,3] obtained a large number of expansions of Appell's double hypergeometric functions. H. M. Srivastava [10] and H. M. Srivastava and P. W. Karlson [12] gave certain interesting integral representations for F_4. Recently M. A. Khan and G. S. Abukhammash [4] introduced 10 Appell's type generalized functions $M_i, i = 1, 2, 3, \cdots, 10$ by considering the product of two ${}_3F_2$ functions instead of product of two Gauss functions taken by Appell to define F_1, F_2, F_3 and F_4 functions. They obtained fractional derivative representations, integral representations symbolic forms and expansion formulae for these newly defined functions $M_1, M_2, \cdots, M_{10}$ similar to those obtained by J. L. Burchnall and T.W. Chaundy for the four Appell's functions.

In the last section of J. L. Burchnall and T. W. Chaundy [2, 3] gave a glimpse of possible extension of their result to functions of higher order (ie, with more parameters) for two variable for instant they defined.

$$
{}_{p+1}F_p^{(2)}\left[\begin{array}{l} a;\ \ b_1, b_2 \cdots\cdots\ b_p; b_1', b_2' \cdots\ b_p'\ ; x, y \\ c_1, c_2 \cdots\cdots\ c_p; c_1', c_2' \cdots\ c_p'\ ; \end{array}\right]
$$

$$
= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m \cdots (b_p)_m\ (b_1')_n(b_2')_n \cdots\ (b_p')_n}{m!n!(c_1)_m(c_2)_m \cdots\cdots\ (c_p)_m(c_1')_n(c_2')_n \cdots\ (c_p')_n}\, x^m\, y^n
$$

$$= \nabla(a)\ {}_{p+1}F_p \left[\begin{array}{l} a, \quad b_1, b_2 \cdots\cdots\ b_p; x \\ c_1, c_2 \cdots\cdots\ c_p; \end{array} \right] p+1F_p \left[\begin{array}{l} a, \quad b_1^{'}, b_2^{'} \cdots\ b_p^{'}; y \\ c_1^{'}, c_2^{'} \cdots\ c_p^{'}; \end{array} \right] \tag{1.1}$$

and gave the result

$${}_{p+1}F_p^{(2)} \left[\begin{array}{l} a; \quad b_1, b_2 \cdots\cdots\ b_p; b_1^{'}, b_2^{'} \cdots\ b_p^{'}\ ; x, y \\ c_1, c_2 \cdots\cdots\ c_p; c_1^{'}, c_2^{'} \cdots\ c_p^{'}\ ; \end{array} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b_1)_r \cdots (b_p)_r\ (b_1^{'})_r (b_2^{'})_r \cdots\ (b_p^{'})_r}{r!(c_1)_r (c_2)_r \cdots\cdots\ (c_p)_r (c_1^{'})_r (c_2^{'})_r \cdots\ (c_p^{'})_r}\, x^r\ y^r$$

$$\times\ {}_{p+1}F_p \left[\begin{array}{l} a+r, \quad b_1+r, b_2+r \cdots\cdots\ b_p+r; x \\ c_1+r, c_2+r \cdots\cdots\ c_p+r; \end{array} \right]$$

$${}_{p+1}F_p \left[\begin{array}{l} a+r, \quad b_1^{'}+r, b_2^{'}+r \cdots\ b_p^{'}+r; y \\ c_1^{'}, c_2^{'} \cdots\ c_p^{'}; \end{array} \right] \tag{1.2}$$

Motivated by this section of [3] and the fact that such functions were encountered by M. A. Khan and G. S. Abukhammash [5, 6] during their study of two variable analogues of Saigo's [9] fractional integral operators, we consider in this paper the product of n-${}_3F_2$ hypergeometric functions viz,

$${}_3F_2(a_1, b_1, c_1; d_1, e_1; x_1)\, {}_3F_2(a_2, b_2, c_2; d_2, e_2; x_2) \cdots\cdots$$

$${}_3F_2(a_n, b_n, c_n; d_n, e_n; x_n) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots\cdots \sum_{m_n=0}^{\infty}$$

$$\frac{(a_1)_{m_1} (a_2)_{m_2} \cdots\ (a_n)_{m_n} (b_1)_{m_1} (b_2)_{m_2} \cdots (b_n)_{m_n} (c_1)_{m_1} (c_2)_{m_2} \cdots (c_n)_{m_n}}{(d_1)_{m_1} (d_2)_{m_2} \cdots (d_n)_{m_n} (e_1)_{m_1} (e_2)_{m_2} \cdots (e_n)_{m_n}}$$

$$\times\ \frac{x_1^{m_1}}{(m_1)!} \cdots\ \frac{x_n^{m_n}}{(m_n)!} \tag{1.3}$$

the n series, in itself, yield nothing new, but if one or more of the each five pairs of products

$(a_1)_{m_1}\cdots(a_n)_{m_n}, (b_1)_{m_1}\cdots(b_n)_{m_n}, (c_1)_{m_1}\cdots(c_n)_{m_n}, (d_1)_{m_1}\cdots(d_n)_{m_n}$, $(e_1)_{m_1}\cdots(e_n)_{m_n}$ be replaced by corresponding expression $(a)_{m_1+m_2+\cdots\ m_n}, (b)_{m_1+m_2+\cdots\ m_n}, (c)_{m_1+m_2+\cdots\ m_n}, (d)_{m_1+m_2+\cdots\ m_n}$, $(e)_{m_1+m_2+\cdots\ m_n}$ we are leading to eleven distinct possibilities of getting new functions. One such possibility, gives us

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots$$

$$\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+m_2+\cdots\ +m_n}\ (b)_{m_1+\cdots\ m_2+\cdots\ m_n}(c)_{m_1+m_2+\cdots+m_n}}{(d)_{m_1+m_2+\cdots+m_n}(c)_{m_1+m_2+\cdots+m_n}}$$

$$\times\ \frac{x_1^{m_1}}{(m_1)!}\cdots\cdots\ \frac{x_n^{m_n}}{(m_n)!}$$

which is simply

$$ {}_3F_2(a,b,c;d,e;x_1+x_2+\cdots\cdots+x_n)$$

Since it is easily verified that (c.f., e.g. H. M. Srivastava [10])

$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots\sum_{m_n=0}^{\infty} f(m_1+m_2+\cdots\cdots+m_n)\frac{x_1^{m_1}}{(m_1)!}\cdots\ \frac{x_n^{m_n}}{(m_n)!}$$

$$= \sum_{N=0}^{\infty}\frac{(x_1+x_2+\cdots\cdots+x_n)^N}{N!} \tag{1.4}$$

The remaining possibilities lead to the ten generalized Appell's type functions of n-variable, which are as defined below subject to suitable convergence conditions:

$$N_1\,(a_1,a_2,\cdots\ a_n,b_1,b_2,\cdots\ b_n,c_1,c_2,\cdots,c_n;d,e_1,e_2,\cdots,e_n;$$

$$x_1,x_2,\cdots\ x_n) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots\sum_{m_n=0}^{\infty}$$

$$\frac{(a_1)_{m_1}(a_2)_{m_2}\cdots\ (a_n)_{m_n}(b_1)_{m_1}(b_2)_{m_2}\cdots(b_n)_{m_n}(c_1)_{m_1}(c_2)_{m_2}\cdots(c_n)_{m_n}}{(d)_{m_1+m_2+\cdots+m_n}(e_1)_{m_1}(e_2)_{m_2}\cdots(e_n)_{m_n}}$$

$$\times\ \frac{x_1^{m_1}}{(m_1)!}\cdots\cdots\ \frac{x_n^{m_n}}{(m_n)!} \tag{1.5}$$

$$N_2(a_1, a_2, \cdots\ a_n, b_1, b_2, \cdots\ b_n, c_1, c_2, \cdots, c_n; d, e; x_1, x_2, \cdots\cdots\ x_n)$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots\ \sum_{m_n=0}^{\infty}$$

$$\frac{(a_1)_{m_1}(a_2)_{m_2}\cdots\ (a_n)_{m_n}(b_1)_{m_1}(b_2)_{m_2}\cdots(b_n)_{m_n}(c_1)_{m_1}(c_2)_{m_2}\cdots(c_n)_{m_n}}{(d)_{m_1+m_2+\cdots+m_n}(e)_{m_1+m_2+\cdots+m_n}}$$

$$\times\ \frac{x_1^{m_1}}{(m_1)!}\cdots\cdots\ \frac{x_n^{m_n}}{(m_n)!} \tag{1.6}$$

$$N_3(a, b_1, b_2, \cdots\ b_n, c_1, c_2, \cdots, c_n; d_1, d_2\cdots, d_n, e_1, e_2, \cdots, e_n; x_1, x_2,$$

$$\cdots\ x_n) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots$$

$$\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+m_2+\cdots+m_n}(b_1)_{m_1}(b_2)_{m_2}\cdots\ (b_n)_{m_n}(c_1)_{m_1}(c_2)_{m_2}\cdots(c_n)_{m_n}}{(d_1)_{m_1}(d_2)_{m_2}\cdots(d_n)_{m_n}(e_1)_{m_1}(e_2)_{m_2}\cdots(e_n)_{m_n}}$$

$$\times\ \frac{x_1^{m_1}}{(m_1)!}\cdots\cdots\ \frac{x_n^{m_n}}{(m_n)!} \tag{1.7}$$

$$N_4(a, b_1, b_2, \cdots\ b_n, c_1, c_2, \cdots, c_n; d, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots\ x_n)$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots$$

$$\sum_{m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n}(b_1)_{m_1}(b_2)_{m_2}\cdots(b_n)_{m_n}(c_1)_{m_1}(c_2)_{m_2}\cdots(c_n)_{m_n}}{(d)_{m_1+m_2+\cdots+m_n}(e_1)_{m_1}(e_2)_{m_2}\cdots(e_n)_{m_n}}$$

$$\times \frac{x_1^{m_1}}{(m_1)!}\cdots\cdots\frac{x_n^{m_n}}{(m_n)!} \tag{1.8}$$

$$N_5(a, b_1, b_2, \cdots b_n, c_1, c_2, \cdots, c_n; d, e; x_1, x_2, \cdots x_n) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots$$

$$\sum_{m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n}(b_1)_{m_1}(b_2)_{m_2}\cdots(b_n)_{m_n}(c_1)_{m_1}(c_2)_{m_2}\cdots(c_n)_{m_n}}{(d)_{m_1+m_2+\cdots+m_n}(e)_{m_1+m_2+\cdots+m_n}}$$

$$\times \frac{x_1^{m_1}}{(m_1)!}\cdots\cdots\frac{x_n^{m_n}}{(m_n)!} \tag{1.9}$$

$$N_6(a, b, c_1, c_2, \cdots, c_n; d_1, d_2 \cdots d_n, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n)$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots$$

$$\sum_{m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n}(b)_{m_1+m_2+\cdots+m_n}(c_1)_{m_1}(c_2)_{m_2}\cdots(c_n)_{m_n}}{(d_1)_{m_1}(d_2)_{m_2}\cdots(d_n)_{m_n}(e_1)_{m_1}(e_2)_{m_2}\cdots(e_n)_{m_n}}$$

$$\times \frac{x_1^{m_1}}{(m_1)!}\cdots\cdots\frac{x_n^{m_n}}{(m_n)!} \tag{1.10}$$

$$N_7(a, b, c_1, c_2, \cdots, c_n; d, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\cdots$$

$$\sum_{m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n}(b)_{m_1+m_2+\cdots+m_n}(c_1)_{m_1}(c_2)_{m_2}\cdots(c_n)_{m_n}}{(d)_{m_1+m_2+\cdots+m_n}(e_1)_{m_1}(e_2)_{m_2}\cdots(e_n)_{m_n}}$$

$$\times \frac{x_1^{m_1}}{(m_1)!} \cdots\cdots \frac{x_n^{m_n}}{(m_n)!} \tag{1.11}$$

$$N_8(a, b, c_1, c_2, \cdots, c_n; d, e; x_1, x_2, \cdots x_n) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots\cdots$$

$$\sum_{m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n} (b)_{m_1+m_2+\cdots+m_n} (c_1)_{m_1} (c_2)_{m_2} \cdots (c_n)_{m_n}}{(d)_{m_1+m_2+\cdots+m_n} (e)_{m_1+m_2+\cdots+m_n}}$$

$$\times \frac{x_1^{m_1}}{(m_1)!} \cdots\cdots \frac{x_n^{m_n}}{(m_n)!} \tag{1.12}$$

$$N_9(a, b, c; d_1, d_2 \cdots, d_n, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots\cdots$$

$$\sum_{m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n} (b)_{m_1+m_2+\cdots+m_n} (c)_{m_1+m_2+\cdots+m_n}}{(d_1)_{m_1} (d_2)_{m_2} \cdots (d_n)_{m_n} (e_1)_{m_1} (e_2)_{m_2} \cdots (e_n)_{m_n}}$$

$$\times \frac{x_1^{m_1}}{(m_1)!} \cdots\cdots \frac{x_n^{m_n}}{(m_n)!} \tag{1.13}$$

$$N_{10}(a, b, c; d, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots\cdots \sum_{m_n=0}^{\infty}$$

$$\frac{(a)_{m_1+m_2+\cdots+m_n} (b)_{m_1+m_2+\cdots+m_n} (c)_{m_1+m_2+\cdots+m_n}}{(d)_{m_1+m_2+\cdots+m_n} (e_1)_{m_1} (e_2)_{m_2} \cdots (e_n)_{m_n}} \times \frac{x_1^{m_1}}{(m_1)!} \cdots \frac{x_n^{m_n}}{(m_n)!} \tag{1.14}$$

2. Symbolic Form

We introduce the inverse pairs of symbolic operators

$$\nabla(h) = \frac{\Gamma(h)\Gamma(\delta_1 + \delta_2 + \cdots + \delta_n + h)}{\Gamma\ (\delta_1 + h)\Gamma\ (\delta_2 + h)\cdots\ \Gamma\ (\delta_n + h)} \tag{2.1}$$

$$\Delta(h) = \frac{\Gamma\ (\delta_1 + h)\Gamma\ (\delta_2 + h)\cdots\ \Gamma\ (\delta_n + h)}{\Gamma(h)\Gamma(\delta_1 + \delta_2 + \cdots + \delta_n + h)} \tag{2.2}$$

where

$$\delta_1 = x_1 \frac{\partial}{\partial\ x_1}, \delta_2 = \ x_2 \frac{\partial}{\partial\ x_2}, \cdots\ \delta_n = x_n \frac{\partial}{\partial\ x_n}$$

then

$$\nabla(h)\ (h)_{m_1}(h)_{m_2}\cdots\ (h)_{m_n}\ x_1^{m_1} x_2^{m_2}\cdots\ x_n^{m_n}\ =\ (h)_{m_1+m_2+\cdots\ m_n}\ x_1^{m_1} x_2^{m_2}$$

$\cdots\ x_n^{m_n}$ and so , if $(h)_{m_1}(h)_{m_2}\cdots\ (h)_{m_n}$ occurs in the numerator of the coefficient of $x_1^{m_1} x_2^{m_2}\cdots\ x_n^{m_n}$, it is changed into $(h)_{m_1+m_2+\cdots\ m_n}$ by the operator $\nabla(h)$. The operator $\Delta(h)$ effects a similar change in the denominator. These symbolic forms were used by them to obtain a large number of expansions of Appell's functions in terms of each other, of Appell's functions in terms of the products of ordinary hypergeometric functions, or vice versa.

In this section we have followed Burhnall and Chaundy's method to obtain the following factorizations of our newly defined functions $N_i, i = 1, 2, 3, ...10$.

$$N_1(a_1, a_2, \cdots\ a_n, b_1, b_2, \cdots\ b_n, c_1, c_2, \cdots, c_n; d, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots$$

$$x_n) = \Delta(d)\ {}_3F_2(a_1, b_1, c_1; d, e_1; x_1){}_3F_2(a_2, b_2, c_2; d, e_2; x_2)\cdots\cdots$$

$${}_3F_2(a_n, b_n, c_n; d, e_n; x_n) \tag{2.3}$$

$$N_2(a_1, a_2, \cdots a_n, b_1, b_2, \cdots b_n, c_1, c_2, \cdots, c_n; d, e; x_1, x_2, \cdots x_n)$$
$$= \Delta(d)\ \Delta(e)\ {}_3F_2(a_1, b_1, c_1; d, e; x_1) {}_3F_2(a_2, b_2, c_2; d, e; x_2) \cdots\cdots$$
$${}_3F_2(a_n, b_n, c_n; d, e; x_n) \tag{2.4}$$

$$N_3(a, b_1, b_2, \cdots b_n, c_1, c_2, \cdots, c_n; d_1, d_2 \cdots, d_n, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots$$
$$x_n) = \nabla(a)\ {}_3F_2(a, b_1, c_1; d, e_1; x_1) {}_3F_2(a, b_2, c_2; d, e_2; x_2) \cdots\cdots$$
$${}_3F_2(a, b_n, c_n; d, e_n; x_n) \tag{2.5}$$

$$N_4(a, b_1, b_2, \cdots b_n, c_1, c_2, \cdots, c_n; d, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n)$$
$$= \nabla(a)\Delta(d)\ {}_3F_2(a, b_1, c_1; d, e_1; x_1) {}_3F_2(a, b_2, c_2; d, e_2; x_2) \cdots\cdots$$
$${}_3F_2(a, b_n, c_n; d, e_n; x_n) \tag{2.6}$$

$$N_5(a, b_1, b_2, \cdots b_n, c_1, c_2, \cdots, c_n; d, e; x_1, x_2, \cdots x_n)$$
$$= \nabla(a)\Delta(d)\Delta(e)\ {}_3F_2(a, b_1, c_1; d, e; x_1) {}_3F_2(a, b_2, c_2; d, e; x_2) \cdots\cdots$$
$${}_3F_2(a, b_n, c_n; d, e; x_n) \tag{2.7}$$

$$N_6(a, b, c_1, c_2, \cdots, c_n; d_1, d_2 \cdots d_n, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n)$$
$$= \nabla(a)\nabla(b)\ {}_3F_2(a, b, c_1; d_1, e_1; x_1) {}_3F_2(a, b, c_2; d_2, e_2; x_2) \cdots\cdots$$
$${}_3F_2(a, b, c_n; d_n, e_n; x_n) \tag{2.8}$$

$$N_7(a,b,c_1,c_2,\cdots,c_n;d,e_1,e_2,\cdots,e_n;x_1,x_2,\cdots x_n)$$

$$= \nabla(a)\nabla(b)\Delta(d)\ {}_3F_2(a,b,c_1;d,e_1;x_1){}_3F_2(a,b,c_2;d,e_2;x_2)\cdots\cdots$$
$${}_3F_2(a,b,c_n;d,e_n;x_n) \tag{2.9}$$

$$N_8(a,b,c_1,c_2,\cdots,c_n;d,e;x_1,x_2,\cdots x_n)$$

$$= \nabla(a)\nabla(b)\Delta(d)\Delta(e)\ {}_3F_2(a,b,c_1;d,e;x_1){}_3F_2(a,b,c_2;d,e;x_2)\cdots\cdots$$
$${}_3F_2(a,b,c_n;d,e;x_n) \tag{2.10}$$

$$N_9(a,b,c;d_1,d_2\cdots d_n,e_1,e_2,\cdots,e_n;x_1,x_2,\cdots x_n)$$

$$= \nabla(a)\nabla(b)\nabla(c)\ {}_3F_2(a,b,c;d,e_1;x_1){}_3F_2(a,b,c;d,e_2;x_2)\cdots\cdots$$
$${}_3F_2(a,b,c;d,e_n;x_n) \tag{2.11}$$

$$N_{10}(a,b,c;d,e_1,e_2,\cdots,e_n;x_1,x_2,\cdots x_n)$$

$$= \nabla(a)\nabla(b)\nabla(c)\Delta(d)\ {}_3F_2(a,b,c;d,e_1;x_1){}_3F_2(a,b,c;d,e_2;x_2)\cdots\cdots$$
$${}_3F_2(a,b,c;d,e_n;x_n) \tag{2.12}$$

$$N_4\left[\begin{array}{c} a,b_1,b_2\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \nabla(a)\ N_1\left[\begin{array}{l} a,a,\cdots,a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right] \tag{2.13}$$

$$N_1\left[\begin{array}{l} a,a,\cdots,a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \Delta(a)\ N_4\left[\begin{array}{c} a,b_1,b_2\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right] \tag{2.14}$$

$$N_5\left[\begin{array}{c} a,b_1,b_2\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \nabla(a)\ N_2\left[\begin{array}{l} a,a,\cdots,a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e; \end{array} x_1,x_2,\cdots,x_n\right] \tag{2.15}$$

$$N_2\left[\begin{array}{l} a,a,\cdots,a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \Delta(a)N_5\left[\begin{array}{c} a,b_1,b_2\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e; \end{array} x_1,x_2,\cdots,x_n\right] \tag{2.16}$$

$$N_4\left[\begin{array}{c} a,b_1,b_2\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \Delta(d)\ N_3\left[\begin{array}{l} a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,d,\cdots,d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right] \tag{2.17}$$

$$N_3\left[\begin{array}{l} a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d,d,\cdots,d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \nabla(d)\ N_4 \left[\begin{array}{c} a, b_1, b_2 \cdots, b_n, c_1, c_2, \cdots, c_n; \\ d, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.18}$$

$$ {}_3F_2 \left[\begin{array}{l} a, b, c; \\ d, e; \end{array} x_1 + x_2 + \cdots + x_n \right]$$

$$= \nabla(c)\ N_8 \left[\begin{array}{l} a, b, c, c, \cdots, c; \\ d, e; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.19}$$

$$ {}_3F_2 \left[\begin{array}{l} a, b, c; \\ d, e; \end{array} x_1 + x_2 + \cdots + x_n \right]$$

$$= \Delta(e)\ N_{10} \left[\begin{array}{l} a, b, c; \\ d, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.20}$$

$$ {}_3F_2 \left[\begin{array}{l} a, b, c; \\ d, e; \end{array} x_1 + x_2 + \cdots + x_n \right]$$

$$= \nabla(c)\Delta(e)\ N_7 \left[\begin{array}{l} a, b, c, c, \cdots, c; \\ d, e, e, \cdots, e; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.21}$$

$$N_6 \left[\begin{array}{l} a, b, c_1, c_2, \cdots, c_n; \\ d_1, d_2, \cdots, d_n, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right]$$

$$= \nabla(b)\ N_3 \left[\begin{array}{c} a, b, b \cdots, b, c_1, c_2, \cdots, c_n; \\ d_1, d_2, \cdots, d_n, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.22}$$

$$N_3 \left[\begin{array}{l} a, b, b, \cdots, b, c_1, c_2, \cdots, c_n; \\ d_1, d_2, \cdots, d_n, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right]$$

$$= \Delta(b)\ N_6 \left[\begin{array}{l} a, b, c_1, c_2, \cdots, c_n; \\ \quad d, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.23}$$

$$N_7 \left[\begin{array}{l} a, b, c_1, c_2, \cdots, c_n; \\ d, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right]$$

$$= \nabla(b)\ N_4 \left[\begin{array}{l} a, b, b \cdots, b, c_1, c_2, \cdots, c_n; \\ \quad d, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.24}$$

$$N_4 \left[\begin{array}{l} a, b, b, \cdots, b, c_1, c_2, \cdots, c_n; \\ d, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right]$$

$$= \Delta(b)\ N_7 \left[\begin{array}{l} a, b, c_1, c_2, \cdots, c_n; \\ \quad d, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.25}$$

$$N_8 \left[\begin{array}{l} a, b, c_1, c_2, \cdots, c_n; \\ d, e; \end{array} x_1, x_2, \cdots, x_n \right]$$

$$= \nabla(b)\ N_5 \left[\begin{array}{l} a, b, b \cdots, b, c_1, c_2, \cdots, c_n; \\ \quad d, e; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.26}$$

$$N_5 \left[\begin{array}{l} a, b, b, \cdots, b, c_1, c_2, \cdots, c_n; \\ d, e; \end{array} x_1, x_2, \cdots, x_n \right]$$

$$= \Delta(b)\ N_8 \left[\begin{array}{l} a, b, c_1, c_2, \cdots, c_n; \\ \quad d, e; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.27}$$

$$N_9 \left[\begin{array}{l} a, b, c; \\ d_1, d_2, \cdots, d_n, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right]$$

$$= \nabla(c)\ N_6 \left[\begin{array}{l} a, b, c, c, \cdots, c; \\ \quad d_1, d_2, \cdots, d_n, e_1, e_2, \cdots, e_n; \end{array} x_1, x_2, \cdots, x_n \right] \tag{2.28}$$

$$N_6\left[\begin{array}{l} a,b,b,\cdots,b,c,c,\cdots,c; \\ d_1,d_2,\cdots,d_n,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \Delta(c)\ N_9\left[\begin{array}{l} a,b,c; \\ \quad d_1,d_2,\cdots,c_n,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right] \tag{2.29}$$

$$N_{10}\left[\begin{array}{l} a,b,c; \\ d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \Delta(c)\ N_7\left[\begin{array}{l} a,b,c,c,\cdots,c; \\ \quad d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right] \tag{2.30}$$

$$N_7\left[\begin{array}{l} a,b,c,c,\cdots,c; \\ d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right]$$

$$= \Delta(c)\ N_{10}\left[\begin{array}{l} a,b,c; \\ \quad d,e_1,e_2,\cdots,e_n; \end{array} x_1,x_2,\cdots,x_n\right] \tag{2.31}$$

3. Fractional Derivates

In 1971 Euiler extended the derivative formula

$$D_z^n\{z^\lambda\} = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)z^{\lambda-1}$$

$$= \frac{\Gamma\ (1+\lambda)}{\Gamma\ (1+\lambda-n)} z^{\lambda-n}(n=0,1,2,3\cdots\cdots\qquad) \tag{3.1}$$

to the general form

$$D_z^\mu\{z^\lambda\} = \frac{\Gamma\ (1+\lambda)}{\Gamma\ (1+\lambda-\mu)} z^{\lambda-\mu} \tag{3.2}$$

where μ is an arbitrary complex number.

Using (3.2) we give the following fractional derivative representations:

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}\ {}_2F_1\left[\begin{array}{l}\alpha_1,\beta_1;\\ \gamma_1\ ;\end{array} x_1 z\right]\ {}_2F_1\left[\begin{array}{l}\alpha_2,\beta_2;\\ \gamma_2\ ;\end{array} x_2 z\right]\cdots\cdots\right.$$

$$\left.{}_2F_1\left[\begin{array}{l}\alpha_n,\beta_n;\\ \gamma_n\ ;\end{array} x_n z\right]\right\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}$$

$$N_4(\lambda,\alpha_1,\alpha_2,\cdots\alpha_n,\beta_1,\beta_2,\cdots,\beta_n;\mu,\gamma_1,\gamma_2,\cdots,\gamma_n;x_1z,x_2z,\cdots x_nz) \tag{3.3}$$

$$D_{y_1}^{b_2-d_2}D_{y_2}^{b_3-d_3}\cdots D_{y_{n-1}}^{b_n-d_n}D_{z_1}^{c_2-e_2}D_{z_2}^{c_3-e_3}\cdots D_{z_{n-1}}^{c_n-e_n}$$

$$\times\ \left\{y_1^{b_2-1}y_2^{b_3-1}\cdots y_{n-1}^{b_n-1}z_1^{c_2-1}z_2^{c_3-1}\cdots z_{n-1}^{c_n-1}(1-y_1z_1-y_2z_2\cdots\cdots\right.$$

$$\left.y_{n-1}z_{n-1})^{-a}\times\ {}_3F_2\left[\begin{array}{l}a,b_1,c_1;\\ d_1,e_1\ ;\end{array}\frac{x}{(1-y_1z_1-y_2z_2\cdots\cdots y_{n-1}z_{n-1})}x_1z\right]\right\}$$

$$=\frac{\Gamma(b_2)\cdots\Gamma(b_n)\Gamma(c_2)\cdots\Gamma(c_n)}{\Gamma(d_2)\cdots\Gamma(d_n)\Gamma(e_2)\cdots\Gamma(e_n)}\ y_1^{d_2-1}y_2^{d_3-1}\cdots y_{n-1}^{d_n-1}z_1^{e_2-1}z_2^{e_3-1}\cdots z_{n-1}^{e_n-1}$$

$$\times\ N_3\left(a,b_1,b_2,\cdots b_n,c_1,c_2,\cdots,c_n;d_1,d_2\cdots,d_n,e_1,e_2,\cdots,e_n;\right.$$

$$\left.x,y_1z_1,y_2z_2,\cdots y_{n-1}z_{n-1}\right) \tag{3.4}$$

$$D_{x_1}^{c_1-e_1}D_{x_2}^{c_2-e_2}\cdots D_{x_n}^{c_n-e_n}\left\{x_1^{c_1-1}x_2^{c_2-1}\cdots x_n^{c_n-1}\right.$$

$$\left.\times\quad F_A^{(n)}\left[a;b_1,b_2,\cdots,b_n;d_1,d_2,\cdots,d_n;x_1,x_2,\cdots,x_n\right]\right\}$$

$$=\frac{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_n)}{\Gamma(e_1)\Gamma(e_2)\cdots\Gamma(e_n)}x_1^{e_1-1}x_2^{e_2-1}\cdots x_n^{e_n-1}\times\ N_3\left(a,b_1,b_2,\cdots\right.$$

$$\left.b_n,c_1,c_2,\cdots,c_n;d_1,d_2\cdots,d_n,e_1,e_2,\cdots,e_n;x_1,x_2,\cdots x_n\right) \tag{3.5}$$

$$D_{x_1}^{c_1-e_1} D_{x_2}^{c_2-e_2} \cdots D_{x_n}^{c_n-e_n} \left\{ x_1^{c_1-1} x_2^{c_2-1} \cdots x_n^{c_n-1} F_D^{(n)} [a; b_1, b_2, \cdots, b_n; d; x_1, x_2, \cdots, x_n] \right\} = \frac{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_n)}{\Gamma(e_1)\Gamma(e_2)\cdots\Gamma(e_n)} x_1^{e_1-1} x_2^{e_2-1} \cdots x_n^{e_n-1} \times N_4(a, b_1, b_2, \cdots b_n, c_1, c_2, \cdots, c_n; d, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n) \tag{3.6}$$

$$D_{x_1}^{c_1-e_1} D_{x_2}^{c_2-e_2} \cdots D_{x_n}^{c_n-e_n} \left\{ x_1^{c_1-1} x_2^{c_2-1} \cdots x_n^{c_n-1} \times F_B^{(n)} [a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n; d; x_1, x_2, \cdots, x_n] \right\} = \frac{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_n)}{\Gamma(e_1)\Gamma(e_2)\cdots\Gamma(e_n)} x_1^{e_1-1} x_2^{e_2-1} \cdots x_n^{e_n-1} \times N_1(a_1, a_2, \cdots, a_n, b_1, b_2, \cdots b_n, c_1, c_2, \cdots, c_n; d, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n) \tag{3.7}$$

$$D_{x_1}^{c_1-e_1} D_{x_2}^{c_2-e_2} \cdots D_{x_n}^{c_n-e_n} \left\{ x_1^{c_1-1} x_2^{c_2-1} \cdots x_n^{c_n-1} F_C^{(n)} [a; b; d_1, d_2, \cdots, d_n; x_1, x_2, \cdots, x_n] \right\} = \frac{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_n)}{\Gamma(e_1)\Gamma(e_2)\cdots\Gamma(e_n)} x_1^{e_1-1} x_2^{e_2-1} \cdots x_n^{e_n-1} \times N_6(a, c_1, c_2, \cdots c_n, c_1, c_2, \cdots, c_n; d_1, d_2 \cdots, d_n, e_1, e_2, \cdots, e_n; x_1, x_2, \cdots x_n) \tag{3.8}$$

$$D_z^{a-d} \left\{ z^{a-1} F_A^{(n)} [b; c_1, c_2, \cdots, c_n; e_1, e_2, \cdots, e_n; x_1 z, x_2 z, \cdots, x_n z] \right\} = \frac{\Gamma(a)}{\Gamma(d)} z^{d-1} N_7(a, b, c_1, c_2, \cdots, c_n; d, e_1, e_2 \cdots, e_n; x_1 z, x_2 z, \cdots x_n z) \tag{3.9}$$

$$D_z^{a-d}\left\{z^{a-1}F_D^{(n)}\left[b;c_1,c_2,\cdots,c_n;e;x_1z,x_2z,\cdots,x_nz\right]\right\}$$

$$=\frac{\Gamma(a)}{\Gamma(d)}z^{d-1}\ N_8(a,b,c_1,c_2,\cdots,c_n;d,e;x_1z,x_2z,\cdots\ x_nz) \tag{3.10}$$

$$D_z^{a-d}\left\{z^{a-1}F_B^{(n)}\left[b_1,b_2,\cdots,b_n;c_1,c_2,\cdots,c_n;e;x_1z,x_2z,\cdots,x_nz\right]\right\}$$

$$=\frac{\Gamma(a)}{\Gamma(d)}z^{d-1}\ N_5(a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n;d,e;x_1z,x_2z,\cdots\ x_nz) \tag{3.11}$$

$$D_z^{a-d}\left\{z^{a-1}F_C^{(n)}\left[b;c;e_1,e_2,\cdots,e_n;x_1z,x_2z,\cdots,x_nz\right]\right\}$$

$$=\frac{\Gamma(a)}{\Gamma(d)}z^{d-1}\ N_7(a,b,c;d,e_1,e_2\cdots,e_n;x_1z,x_2z,\cdots\ x_nz) \tag{3.12}$$

$$D_z^{a-d}\left\{z^{a-1}F_B^{(n)}\left[a_1,a_2,\cdots,a_n,b_1,b_2,\cdots,b_n;c_1,c_2,\cdots,c_n;\right.\right.$$

$$\left.\left.d,e_1,e_2,\cdots,e_n;x_1z,x_2z,\cdots,x_nz\right]\right\}=\frac{\Gamma(a)}{\Gamma(d)}z^{d-1}N_1\left(a_,a_2,\cdots,a_n,\right.$$

$$\left.b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n;d,e_1,e_2\cdots,e_n;x_1z,x_2z,\cdots\ x_nz\right) \tag{3.13}$$

$$D_z^{a-d_1}D_{w_1}^{d_1-d_2}D_{w_2}^{d_1-d_3}\cdots\ D_{w_{n-1}}^{d_1-d_n}\left\{{w_1}^{d_1-1}{w_2}^{d_1-1}\cdots\ {w_{n-1}}^{d_1-1}z^{a-1}\right.$$

$$\left.\times\ \nabla(d_1)F_C^{(n)}\left[b;c;e_1,e_2,\cdots,e_n;x_1z,w_1x_2z,w_2x_2z\cdots,w_{n-1}x_nz\right]\right\}$$

$$=\frac{\Gamma(a)}{\Gamma(d_1)\Gamma(d_2)\cdots\ \Gamma(d_n)}z^{d_1-1}\ N_9\left(a,b,c;d_1,d_2,\cdots,d_n,e_1,e_2\cdots,e_n;\right.$$

$$\left.x_1z,w_1x_2z,\cdots\ w_{n_1}x_nz\right) \tag{3.14}$$

$$D_z^{a_1-d} D_{w_1}^{a_2-a_1} D_{w_2}^{a_3-a_1} \cdots\ D_{w_n}^{a_n-a_1} \left\{ {w_1}^{a_2-1} {w_2}^{a_3-1} \cdots\ {w_n}^{a_n-1} z^{a_1-1} \right.$$

$$\left. \times\ \Delta(a_1) F_D^{(n)} \left[\begin{array}{l} b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n; \\ \quad e; \end{array} x_1 z, w_1 x_2 z, \cdots, w_{n-1} x_n z \right] \right\}$$

$$= \frac{\Gamma(a)}{\Gamma(d_1)\Gamma(d_2)\cdots\ \Gamma(d_n)} \times z^{d_1-1}\ N_2\left(a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n,\right.$$

$$\left. c-1, c_2, \cdots, c_n; d, e; x_1 z, w_1 x_2 z, \cdots\ w_{n-1} x_n z\right) \qquad (3.15)$$

4. Integral Representations

In the theory of Euilerian integrals, the elementary formulas

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, Re(\alpha) > 0, Re(\beta) > 0 \qquad (4.1)$$

$$\int\int u^{\alpha-1} v^{\beta-1} (1-u-v)^{\gamma-1} du dv = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)}$$

$$u \geq 0, v \geq 0, u+v \leq\ 1, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0 \qquad (4.2)$$

and so,

$$\int\int \cdots\cdots \int u_1^{\alpha_1-1} u_2^{\alpha_2-1} \cdots\cdots\ u_n^{\alpha_n-1} (1-u_1-u_2-\cdots\cdots\ -u_n)^{\beta-1}$$

$$du_1 du_2 \cdots\ du_n = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\cdots\Gamma(\alpha_n)\Gamma(\beta)}{\Gamma(\alpha_1+\alpha_2+\cdots\ \alpha_n+\beta)}$$

$$u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0, u_1+u_2+\cdots+u_n \leq\ 1,$$

$$Re(\alpha_1) > 0, Re(\alpha_2) > 0, \cdots, Re(\alpha_n) > 0, Re(\beta) > 0 \qquad (4.3)$$

Making use of (4.1) and (4.3) we give the following integral representations for

$N_1, N_2, ..., N_{10}$

$$\int\int \cdots\cdots \int u_1^{c_1-1} u_2^{c_2-1} \cdots\cdots u_n^{c_n-1}$$

$$(1-u_1-u_2-\cdots\cdots -u_n)^{d-c_1-c_2-\cdots -c_n-1} du_1 du_2 \cdots du_n$$

$$\times {}_2F_1\left[\begin{array}{l} a_1, b_1; \\ e_1\ ; \end{array} u_1 x_1\right] {}_2F_1\left[\begin{array}{l} a_2, b_2; \\ e_2\ ; \end{array} u_2 x_2\right] \cdots\cdots 2F_1\left[\begin{array}{l} a_n, b_n; \\ e_n\ ; \end{array} u_n\ x_n\right]$$

$$du_1\ du_2 \cdots du_n = \frac{\Gamma(c_1)\Gamma(c_2)\cdots\cdots\Gamma(c_n)\Gamma(d-c_1-c_2-\cdots -c_n)}{\Gamma(d)}$$

$$\times N_1\,(a_, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n; d, e_1, e_2 \cdots, e_n;$$

$$x_1, x_2, \cdots x_n) \tag{4.4}$$

$$u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0, u_1 + u_2 + \cdots + u_n \leq\ 1, Re(c_1) > 0,$$

$$Re(c_2) > 0, \cdots, Re(c_n) > 0, Re(d - c_1 - c_2 - \cdots\ c_n) > 0$$

$$\frac{\Gamma(c_1)\Gamma(c_2)\cdots\cdots\Gamma(c_n)}{\Gamma(c_1+c_2+\cdots\ +c_n)} \times\ N_1\,(a_, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n;$$

$$c_1 + c_2 + \cdots\ + c_n, d_1, d_2 \cdots, d_n; x_1, x_2, \cdots x_n)$$

$$= \int_0^1 u^{c_1-1}(1-u)_{c_2-1}(1-u)^{c_3-1} \cdots\cdots (1-u)^{c_n-1} \times {}_2F_1\left[\begin{array}{l} a_1, b_1; \\ d_1\ ; \end{array} ux_1\right]$$

$${}_2F_1\left[\begin{array}{l} a_2, b_2; \\ d_2\ ; \end{array} (1-u)x_2\right] \cdots\cdots 2F_1\left[\begin{array}{l} a_n, b_n; \\ d_n\ ; \end{array} (1-u)\ x_n\right] du \tag{4.5}$$

$$Re(c_1) > 0, Re(c_2) > 0, \cdots, Re(c_n) > 0$$

$$\int\int \cdots\cdots \int u_1^{c_1-1} u_2^{c_2-1} \cdots\cdots u_n^{c_n-1}$$

$$(1-u_1-u_2-\cdots\cdots -u_n)^{d-c_1-c_2-\cdots -c_n-1} du_1 du_2 \cdots du_n$$

$$\times F_B^n \left[\begin{array}{l} a_1, a_2, \cdots a_n, b_1, b_2, \cdots, b_n; \\ e\ ; \end{array} u_1\ x_1, u_2 x_2 \cdots, u_n x_n \right]$$

$$du_1\, du_2 \cdots du_n = \frac{\Gamma(c_1)\Gamma(c_2)\cdots\cdots\Gamma(c_n)\Gamma(d-c_1-c_2-\cdots -c_n)}{\Gamma(d)}$$

$$\times N_2(a_, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n; d, e; x_1, x_2, \cdots x_n) \tag{4.6}$$

$$u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0, u_1+u_2+\cdots+u_n \leq\ 1, Re(c_1) > 0,$$

$$Re(c_2) > 0, \cdots, Re(c_n) > 0, Re(d-c_1-c_2-\cdots\ c_n) > 0$$

$$\frac{\Gamma(c_1)\Gamma(c_2)\cdots\cdots\Gamma(c_n)}{\Gamma(c_1+c_2+\cdots\ +c_n)} \times\ N_2\,(a_, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n;$$

$$c_1+c_2+\cdots\ +c_n, d_1, d_2 \cdots, d_n; x_1, x_2, \cdots\ x_n)$$

$$= \int_0^1 u^{c_1-1}(1-u)_{c_2-1}(1-u)^{c_3-1}\cdots\cdots(1-u)^{c_n-1}$$

$$\times F_B^n \left[\begin{array}{l} a_1, a_2, \cdots a_n, b_1, b_2, \cdots, b_n; \\ d\ ; \end{array} u\ x_1, (1-u)x_2 \cdots, (1-u)x_n \right] du \tag{4.7}$$

$$Re(c_1) > 0, Re(c_2) > 0, \cdots, Re(c_n) > 0$$

$$\frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_n)\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_n)\Gamma(d_1-b_1)\Gamma(d_2-b_2)\cdots\Gamma(d_n-b_n)}{\Gamma(e_1)\Gamma(e_2)\cdots\Gamma(e_n)}$$

$$\times\ \frac{\Gamma(e_1-c_1)\Gamma(c_2-e_2)\cdots\Gamma(e_n-c_n)}{\Gamma(d_1)\Gamma(d_2)\cdots\Gamma(d_n)}$$

$$N_3\left[\begin{array}{l} a, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n; \\ d_1, d_2 \cdots, d_n, e_1, e_2, \cdots, e_n \; ; \end{array} x_1, x_2 \cdots, x_n\right]$$

$$= \int\int \cdots \int u_1^{b_1-1} u_2^{b_2-1} \cdots u_n^{b_n-1} v_1^{c_1-1} v_2^{c_2-1} \cdots\cdots v_n^{e_n-1})$$

$$\times (1-u_1)_{d_1-b_1-1}(1-u_2)^{d_2-b_2-1} \cdots (1-u_n)^{d_n-b_n-1}$$

$$\times (1-v_1)_{e_1-c_1-1}(1-v_2)^{e_2-c_2-1} \cdots (1-v_n)^{e_n-c_n-1}$$

$$\times (1-u_1v_1x_1-u_2v_2x_2-\cdots\cdots-u_nv_nx_n)^{-a} dv_1 dv_2 \cdots dv_n \, du_1 du_2 \cdots du_n \tag{4.8}$$

$$Re(d_1) > Re(b_1) > o, Re(d_2) > Re(b_2) > 0, \cdots, Re(d_n) > Re(b_n) > 0,$$

$$Re(e_1) > Re(e_2) > 0, \cdots, Re(e_n) > Re(c_n) > 0$$

$$\frac{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_n)\Gamma(d-c_1-c_2-\cdots-c_n)}{\Gamma(d)}$$

$$\times N_4\left[\begin{array}{l} a, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n; \\ d \;, e_1, e_2, \cdots, e_n; \end{array} ; x_1, x_2, \cdots x_n)\right]$$

$$= \int\int \cdots\cdots \int u_1^{c_1-1} u_2^{c_2-1} \cdots\cdots u_n^{c_n-1}$$

$$(1-u_1-u_2-\cdots-u_n)^{d-c_1-c_2-\cdots-c_n-1}$$

$$\times F_A^n\left[\begin{array}{l} a, b_1, b_2, \cdots, b_n; \\ c_1, c_2, \cdots, c_n \; ; \end{array} u\, x_1, ux_2 \cdots, ux_n\right] du_1\, du_2 \cdots du_n \tag{4.9}$$

$$u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0, u_1+u_2+\cdots+u_n \leq 1, Re(c_1) > 0,$$

$$Re(c_2) > 0, \cdots, Re(c_n) > 0, Re(d-c_1-c_2-\cdots c_n) > 0$$

$$\frac{\Gamma(c_1)\Gamma(c_2)\cdots\cdots\Gamma(c_n)\Gamma(d-c_1-c_2-\cdots\ -c_n)}{\Gamma(d)}$$

$$\times N_5\left[\begin{array}{l} a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d\ ,e; \end{array} ;x_1,x_2,\cdots\ x_n)\right]$$

$$= \int\int\cdots\cdots\ \int u_1^{c_1-1}u_2^{c_2-1}\cdots\cdots\ u_n^{c_n-1}$$

$$(1-u_1-u_2-\cdots-u_n)^{d-c_1-c_2-\cdots-c_n-1}$$

$$\times F_D^n\left[\begin{array}{l} a,b_1,b_2,\cdots,b_n; \\ e\ ; \end{array} u_1\ x_1,u_2x_2\cdots,u_nx_n\right]du_1\ du_2\ \cdots\ du_n \tag{4.10}$$

$$u_1\geq 0,u_2\geq 0,\cdots,u_n\geq 0,u_1+u_2+\cdots+u_n\leq\ 1,Re(c_1)>0,$$

$$Re(c_2)>0,\cdots,Re(c_n)>0,Re(d-c_1-c_2-\cdots\ c_n)>0$$

$$\frac{\Gamma(c_1)\Gamma(c_2)\cdots\cdots\Gamma(c_n)\Gamma(d_1-c_1)\Gamma(d_2-c_2)\cdots\cdots\Gamma(d_n-c_n)}{\Gamma(d_1)\Gamma(d_2)\cdots\cdots\Gamma(d_n)}$$

$$\times N_6\left[\begin{array}{l} a,b_1,b_2,\cdots,b_n,c_1,c_2,\cdots,c_n; \\ d_1,d_2,\cdots,d_n\ ,e_1,e_2,\cdots,e_n; \end{array} ;x_1,x_2,\cdots\ x_n)\right]$$

$$\int_0^1\int_0^1\cdots\cdots\ \int_0^1 u_1^{c_1-1}u_2^{c_2-1}\cdots\cdots\ u_n^{c_n-1}(1-u_1)^{d_1-c_1-1}$$

$$(1-u_2)^{d_2-c_2-1}\cdots\ (1-u_n)^{d_n-c_n-1}$$

$$\times F_C^n\left[\begin{array}{l} a,b; \\ e_1,e_2,\cdots,e_n\ ; \end{array} u_1\ x_1,u_2x_2\cdots,u_nx_n\right]du_1\ du_2\ \cdots\ du_n \tag{4.11}$$

$$Re(d_1)>Re(c_1)>o,Re(d_2)>Re(c_2)>0,\cdots,Re(d_n)>Re(c_n)>0$$

$$\frac{\Gamma(c_1)\Gamma(c_2)\cdots\cdots\Gamma(c_n)\Gamma(1-c_1-c_2-\cdots\ -c_n)}{\Gamma(d)}$$

$$N_7\left[\begin{array}{l} a, b, c_1, c_2, \cdots, c_n; \\ d\ , e_1, e_2, \cdots, e_n; \end{array} ; x_1, x_2, \cdots\ x_n)\right]$$

$$= \int\int \cdots\cdots \int u_1^{c_1-1} u_2^{c_2-1} \cdots\cdots\ u_n^{c_n-1}$$

$$(1 - u_1 - u_2 - \cdots - u_n)^{d-c_1-c_2-\cdots-c_n-1}$$

$$\times F_C^n\left[\begin{array}{l} a, b; \\ e_1, e_2, \cdots, e_n\ ; \end{array}\ u_1\ x_1, u_2 x_2 \cdots, u_n x_n\right] du_1\ du_2\ \cdots\ du_n \quad (4.12)$$

$$u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0, u_1 + u_2 + \cdots + u_n \leq\ 1, Re(c_1) > 0,$$

$$Re(c_2) > 0, \cdots, Re(c_n) > 0, Re(d - c_1 - c_2 - \cdots\ c_n) > 0$$

$$\frac{\Gamma(c_1)\Gamma(c_2)\cdots\cdots\Gamma(c_n)\Gamma(d - c_1 - c_2 - \cdots\ - c_n)}{\Gamma(d)}$$

$$\times\ N_8\left[\begin{array}{l} a, b, c_1, c_2, \cdots, c_n; \\ d\ , e; \end{array} ; x_1, x_2, \cdots\ x_n)\right]$$

$$= \int\int \cdots\cdots \int u_1^{c_1-1} u_2^{c_2-1} \cdots\cdots\ u_n^{c_n-1}$$

$$(1 - u_1 - u_2 - \cdots - u_n)^{d-c_1-c_2-\cdots-c_n-1}$$

$$\times\ {}_2F_1\ \left[\begin{array}{l} a, b; \\ e\ ; \end{array}\ u_1\ x_1 + u_2 x_2 + \cdots + u_n x_n\right] du_1\ du_2\ \cdots\ du_n \quad (4.13)$$

$$u_1 \geq 0, u_2 \geq 0, \cdots, u_n \geq 0, u_1 + u_2 + \cdots + u_n \leq\ 1, Re(c_1) > 0,$$

$$Re(c_2) > 0, \cdots, Re(c_n) > 0, Re(d - c_1 - c_2 - \cdots\ c_n) > 0$$

$$\frac{\Gamma(c_1)\Gamma(d-c_2)\cdots\cdots\Gamma(d-c_n)}{\Gamma(d)}$$

$$\times\ N_{10}\left[\begin{array}{l} a,b,c_1; \\ d\ ,e_1,e_2,\cdots,e_n; \end{array} ;x_1,x_2,\cdots\ x_n)\right]$$

$$= \int_0^1 u^{c_1-1}(1-u)^{d-c_1-1} F_C^n\left[\begin{array}{l} a,b; \\ e_1,e_2,\cdots,e_n\ ; \end{array} u_1\ x_1,u_2x_2\cdots,u_nx_n\right] du \tag{4.14}$$

$$Re(d) > Re(c_1) > 0$$

References

[1] Appell P.: *Sur une classe de polynomes.* Ann. Sci. Ecole Norm. Sup. (2), Vol. 9, pp. 119-144, 1980.

[2] Burchnall, J. L. and Chaundy, T. W.: *Expansions of Appell's double hypergeometric functions.* Quart. J. Math. Oxford Ser., Vol. 11, pp. 249-270, 1940.

[3] Burchnall, J. L. and Chaundy, T. W.: *Expansions of Appell's double hypergeometric functions(II).* Quart. J. Math. Oxford Ser., Vol. 12, pp. 112-128, 1941.

[4] Khan, M. A. and Abukhmmash, G. S.: *On a generalizations of Appell's functions of two variables.* Pro Mathematica Vol XVI, Nos. 31-32, pp. 61-83, 2002.

[5] Khan, M. A. and Abukhmmash, G. S.: *A study of two variable analogues of certain fractional integral operators.* Pro Mathematica Vol XVII, Nos. 33, pp., 2003.

[6] Khan, M. A. and Abukhmmash, G. S.: *On a certain fractional integral operators of two variable and integral transform.* Pro Mathematica Vol XVII, Nos. 34, pp., 2003.

[7] Lauricella, G.: *Sulle funzioni ipergeometriche a piu variabili.* Rend. Circ. Mat. palermo, Vol. 7, pp. 111-158, 1893.

[8] Rainville, E. D.: *Special Functions.* Macmillan, New York; Reprinted by Chelsea Publ. Co., Bronx., New York, (1971).

[9] Saigo, M.: *A Remark on integral operators involving Hypergeometric functions.* math. Rep. kuyshu Univ. Vol. 11, pp. 135-143, 1978.

[10] Srivastava, H. M.: *Certain double integrals involving hypergeometric functions.* Jnanbha Sect. A., vol. 1, pp. 1-10, 1984.

[11] Srivastava, H. M. and Manocha, H. L.: *A Treatise on Generating Functions.* John Wiley & Sons (Halsted Press), New York; Ellis Horwood, Chichester, 1984.

[12] Srivastava, H. M. and Karlson, P. W.: *Multipl Gaussian Hypergeometric series.* Ellis Horwood Limited, Chichester, (1985)

Resumen

El presente artículo introduce 10 tipo de funciones generalizadas tipo Appell N_i, $1 \leq i \leq 10$, considerando el producto de n funciones ${}_3F_2$. El artículo contiene representaciones por derivadas fraccionales, representaciones integrales y formas simbólicas similares a aquellas obtenidas por J. L. Burchnall y T. W. Chaundy para las cuatro funciones de Appell, han sido obtenidas para estas nuevas funciones N_1, $N_2 \ldots\ldots N_{10}$. Los resultados parecen ser nuevos.

Palabras Clave: Series hipergeométricas, funciones de Lauricella, derivadas fraccionales y representaciones integrales.

Mumtaz Ahmad Khan
mumtaz_ahmad_khan_2008@yahoo.com

K. S. Nisar
ksnisar1@gmail.com

Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University.
Aligarh - 202002, U.P., India.