SOME PROPERTIES OF THE BEURLING CORRELATION FUNCTION

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Abstract

We review properties of the Beurling correlation function related to differentiability and functional equations. The relevance of this function is due to the fact that some properties of the Riemann zeta function can be expressed in terms of it.

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In 1955 A. Beurling proved the following theorem [5]

Theorem 1. *If*

$$M = \{f | f(x) = \sum_{k=1}^{N} a_k \rho\left(\frac{\theta_k}{x}\right), \sum_{k=1}^{N} a_k\theta_k = 0, 0 < \theta_k \leq 1, a_k \in \mathbb{C}, \quad 1 \leq k \leq N, N \geq 2\}$$

where $\rho(x) = x - [x]$, is the fractionary part function, then the Riemann Hypothesis (RH) holds iff $\overline{M} = L^2(0,1)$. Moreover this last condition holds iff $1 \in \overline{M}$.

In trying to verify the condition $1 \in \overline{M}$, one comes across the following integrals

$$\int_0^1 \rho\left(\frac{\theta}{x}\right) dx = -\theta \ln\theta + (1-\gamma)\theta, \ \theta \in [0,1],$$

$$\int_0^1 \rho\left(\frac{\theta}{x}\right)^2 dx = \left(\ln(2\pi) - \gamma\right)\theta - \theta^2, \ \theta \in [0,1],$$

$$\int_0^1 \rho\left(\frac{\alpha}{x}\right)\rho\left(\frac{\beta}{x}\right) dx = \beta\int_0^1 \rho\left(\frac{1}{x}\right)\rho\left(\frac{\frac{\alpha}{\beta}}{x}\right) dx + (1-\beta)\alpha$$

where $0 < \alpha \leq \beta \leq 1$ and γ is Euler constant.

The Beurling correlation function is defined by

$$J(\beta) = \int_0^1 \rho\left(\frac{1}{x}\right)\rho\left(\frac{\beta}{x}\right) dx, \ \beta \in [0,1] \tag{1}$$

In [4] it is studied the closely related function

$$A(\lambda) = \int_0^\infty \rho\left(t\right)\rho\left(\lambda t\right) t^{-2} dt, \ \lambda > 0 \tag{2}$$

that is called the multiplicative correlation function for the fractionary part function. It is not difficult to show that

$$A(\lambda) = \begin{cases} \lambda + J(\lambda) & \text{if} \quad \lambda \in [0,1] \\ 1 + \lambda J(\lambda^{-1}) & \text{if} \quad \lambda > 1 \end{cases} \tag{3}$$

It is proven in [4] that A is absolutely continuous, has a strict local maximum at each point in $\mathbb{Q} \cap]0,1[$, and it is not differentiable in this set.

If in (1) we assume that $\beta > 1$, it can be shown that

$$J(\beta) = \beta J(\beta^{-1}) + 1 - \beta + [\beta] \ln \beta - \ln [\beta]!, \; \beta > 1 \tag{4}$$

where $[\beta]$ is the greatest integer $\leq \beta$.

In [2] from the fact that the function

$$\rho\left(\frac{\alpha}{x}\right) + \rho\left(\frac{1-\alpha}{x}\right) - \rho\left(\frac{1}{x}\right)$$

only takes the values 0 and 1, it is proven that J obeys the following functional equation

$$-\frac{\alpha \ln \alpha}{2} - \frac{(1-\alpha)\ln(1-\alpha)}{2} = J(1) - J(\alpha) - J(1-\alpha) + \alpha +$$

$$+(1-\alpha)J(\frac{\alpha}{1-\alpha}), \; \forall \alpha \in \left[0, \frac{1}{2}\right] \tag{5}$$

In a similar way from the fact that the function

$$\rho\left(\frac{\alpha}{2x}\right) + \rho\left(\frac{\beta}{2x}\right) + \rho\left(\frac{\alpha+\beta}{2x}\right) - \rho\left(\frac{\alpha}{x}\right) - \rho\left(\frac{\beta}{x}\right)$$

only takes the values 0 and 1 we get the functional equations

$$\frac{1}{2}[\ln 2+3\ln\pi-3\gamma]+4(1-\alpha)J\left(\frac{\alpha}{1-\alpha}\right)+2\alpha^2$$
$$=1+2(1-\alpha)J\left(\frac{\alpha}{2(1-\alpha)}\right)+$$
$$+(1-\alpha)J\left(\frac{2\alpha}{1-\alpha}\right)+\alpha(\alpha+1)+J(2\alpha)+2(1-\alpha)J\left(\frac{1}{2(1-\alpha)}\right),$$
$$\forall\alpha\in\left[0,\frac{1}{3}\right] \tag{6}$$

$$\frac{1}{2}[2\ln 2+3\ln\pi-3\gamma]+4(1-\alpha)J\left(\frac{\alpha}{1-\alpha}\right)+2\alpha^2$$
$$=1+2(1-\alpha)J\left(\frac{\alpha}{2(1-\alpha)}\right)+$$
$$+2\alpha J\left(\frac{1-\alpha}{2\alpha}\right)+(1-\alpha)^2+J(2\alpha)+2(1-\alpha)J\left(\frac{1}{2(1-\alpha)}\right),$$
$$\forall\alpha\in\left[\frac{1}{3},\frac{1}{2}\right] \tag{7}$$

A closed formula for J is

$$J(\alpha)=-\frac{\alpha\ln\alpha}{2}-\sum_{n\geq 1}\left\{\left(\sum_{k=1}^{[\frac{n}{\alpha}]}\frac{1}{k}\right)-\ln\left[\frac{n}{\alpha}\right]-\gamma-\frac{1}{2[\frac{n}{\alpha}]}\right\}+$$
$$-\sum_{n\geq 1}\left\{\ln\left(1-\frac{\alpha\rho\left(\frac{n}{\alpha}\right)}{n}\right)+\frac{\alpha\rho\left(\frac{n}{\alpha}\right)}{n}\right\}-\frac{1}{2}\sum_{n\geq 1}\frac{\alpha\rho\left(\frac{n}{\alpha}\right)}{\left[\frac{n}{\alpha}\right]n}+$$
$$+\frac{\alpha}{2}\left\{\ln(2\pi)-\gamma-1\right\},\ \forall\alpha\in[0,1] \tag{8}$$

From (8) we can get the behaviour of J for small α:

$$J(\alpha) = -\frac{\alpha\ln\alpha}{2} + \frac{\alpha}{2}\{\ln(2\pi) - \gamma - 1\} + \\ + \alpha^2\left\{\frac{\pi^2}{72} + \frac{1}{2}\sum_{n\geq 1}\frac{\rho\left(\frac{n}{\alpha}\right)^2}{n^2} - \frac{1}{2}\sum_{n\geq 1}\frac{\rho\left(\frac{n}{\alpha}\right)}{n^2}\right\} + \\ +\alpha^3\left\{\frac{1}{6}\sum_{n\geq 1}\frac{\rho\left(\frac{n}{\alpha}\right)}{n^3} + \frac{1}{3}\sum_{n\geq 1}\frac{\rho\left(\frac{n}{\alpha}\right)^3}{n^3} - \frac{1}{2}\sum_{n\geq 1}\frac{\rho\left(\frac{n}{\alpha}\right)^2}{n^3}\right\} + O(\alpha^4) \quad (9)$$

If μ is the Möbius function it can be shown that [7]

$$\sum_{n\geq 1}\mu(n)\left[\frac{x}{n}\right] = 1, \quad \forall x \geq 1 \quad (10)$$

and

$$\sum_{n\geq 1}\frac{\mu(n)}{n} = 0 \quad (11)$$

From these last two equations it follows that

$$\sum_{n\geq 1}\mu(n)\rho\left(\frac{\theta}{nx}\right) = -\chi_{]0.\theta]}(x), \quad \forall\theta, x \in]0,1] \quad (12)$$

It can be shown that this series (θ fixed, x variable) does not converge in $L^2(0,1)$, but it does in $L^1(0,1)$, [3], theorem 2.2. Therefore one can multiply (12) by $\rho\left(\frac{1}{x}\right)$ and integrate from 0 to 1 to get

$$\sum_{n\geq 1}^{\infty}\mu(n)J\left(\frac{\theta}{n}\right) = \gamma - \ln\theta - \sum_{n=1}^{\left[\frac{1}{\theta}\right]}\frac{1}{n} - \theta\rho\left(\frac{1}{\theta}\right) \quad (13)$$

The series in (13) converges since [9], p.74

$$\sum_{n\geq 1} \frac{\mu(n)}{n} \ln n = -1 \tag{14}$$

To find the distributional derivative of J it is convenient to rewrite (8) as

$$J(\alpha) = -\frac{\alpha \ln \alpha}{2} + \frac{\alpha}{2}\{\ln(2\pi) - \gamma - 1\} - \sum_{n\geq 1} \int_0^{\frac{\alpha}{n}} \left\{\rho\left(\frac{1}{t}\right) - \frac{1}{2}\right\} dt$$

$$\forall \alpha \in [0,1] \tag{15}$$

From the formula

$$\int_0^{\frac{\alpha}{n}} \left\{\rho\left(\frac{1}{t}\right) - \frac{1}{2}\right\} dt = 1 - \gamma - \ln\left(\frac{n}{\alpha}\right) + \sum_{l=1}^{\left[\frac{n}{\alpha}\right]} \frac{1}{l} - \frac{\alpha}{n}\left[\frac{n}{\alpha}\right] - \frac{\alpha}{2n} \tag{16}$$

it can be shown that the right hand side is differentiable if and only if $\frac{n}{\alpha} \notin \mathbb{N}$ and the value of the derivative is $\frac{\rho\left(\frac{n}{\alpha}\right) - \frac{1}{2}}{n}$, but if $\frac{n}{\alpha} \in \mathbb{N}$ the value of the left hand derivative is $-\frac{1}{2n}$ and of the right hand derivative is $\frac{1}{2n}$. Therefore for α irrational in $[0,1]$ the distributional derivative of the Beurling correlation function is

$$J'(\alpha) = \frac{1}{2}\{\ln(2\pi) - \gamma - 2\} - \frac{\ln \alpha}{2} - \sum_{n\geq 1} \frac{\rho\left(\frac{n}{\alpha}\right) - \frac{1}{2}}{n} \tag{17}$$

Now it is known that for an absolutely continuous function the distributional derivative and the usual derivative coincide almost everywhere [11], p.5, corollary 1.1. We conjecture that the ordinary derivative of J coincides with (17) at every point where the series

$$\sum_{n\geq 1} \frac{\rho\left(\frac{n}{\alpha}\right) - \frac{1}{2}}{n} \tag{18}$$

converges. Fortunately the convergence of the series (18) has been studied in the literature [6], [8], where it is proven that if $\{q_j(x)\}_{j\geq 1}$ is the sequence of denominators of the convergents of the continued fraction expansion of the irrational number x, then the series

$$\sum_{n\geq 1} \frac{\rho\,(nx) - \frac{1}{2}}{n} \tag{19}$$

converges if and only if the series

$$\sum_{j\geq 1} \frac{(-1)^j \ln q_{j+1}(x)}{q_j(x)} \tag{20}$$

converges.

It is shown in [8] that the Hausdorff dimension of the divergence set of (20) is equal to zero. If we derive (5) and replace (17) in the resulting equation we get

$$J(\beta) = \left(\frac{1-\beta}{2}\right)\ln\beta - \beta + \left(\frac{1+\beta}{2}\right)\{\ln(2\pi) - \gamma\} +$$
$$-\beta \sum_{m\geq 1} \frac{\rho\left(\frac{m}{\beta}\right) - \frac{1}{2}}{m} - \sum_{m\geq 1} \frac{\rho\,(m\beta) - \frac{1}{2}}{m} \tag{21}$$

This formula can also be proven rigourosly by other method if we assume the convergence of both series.

In [1] it is proven the following theorem

Theorem 2. *Let*

$$[A_\rho f](\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)\,dx$$

be considered as an operator on $L^2(0,1)$*. Then RH holds iff* $KerA_\rho = 0$ *or iff* $h \notin R(A_\rho)$*, where* $h(x) = x, \forall x \in [0,1]$*.*

By a result of Sebestyén [10], $h \in R(A_\rho)$ iff there is a constant $m_h > 0$ such that

$$| < \phi, h > | \leq m_h \|A_\rho^* \phi\|, \ \forall \phi \in L^2(0,1) \tag{22}$$

If ϕ is real one can show that

$$\|A_\rho^* \phi\|^2 = 2 \int_0^1 \int_0^u \phi(u)\phi(v) \left\{ uJ\left(\frac{v}{u}\right) + v - uv \right\} dv\, du \tag{23}$$

Therefore RH holds iff one can find a sequence $\{\phi_n\}_{n\geq 1}$ in $L^2(0,1)$ such that

$$lim \frac{| < \phi_n, h > |}{\|A_\rho^* \phi_n\|} = \infty \tag{24}$$

The relation (24) could be amenable to numerical experimentation.

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Resumen

Se repasan algunas propiedades de la función de correlación de Beurling, que sirven para expresar ciertas propiedades de la función zeta de Riemann.

Palabras clave: Función de correlación de Beurling, ecuaciones funcionales y condición diofántica para diferenciabilidad.

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