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MULTIPLY HÖLDER FUNCTIONS[a]

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(*Presentado por R. Rabanal*)

Abstract

In this article we show several properties about multiply Hölder functions. We study the Hölder class of a composition of multiply Hölder functions and prove that a map and its inverse belong — under certain hypotheses — to the same Hölder class. We also prove some extension properties of multiply Hölder functions; for example, we show that a multiply Hölder functions always extends, in the same Hölder class, to "exceptional" sets that are codimension one manifolds.

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1. Introduction

Hölder functions are a classic topic in analysis with many applications in other areas, as the theory of partial differential equations and dynamical systems. However, besides the restricted treatment given in books of partial differential equations –see for example [2, 4]– a systematic study of the subject is still incomplete; the best reference in this sense is the book of R. Fiorenza [3].

Let $\alpha \in (0,1]$ and $U \subset \mathbb{R}^m$. A function $f\colon U \to \mathbb{R}^n$ is called α-Hölder if the set

$$\left\{ \frac{|f(x)-f(y)|}{|x-y|^\alpha} \colon x, y \in U,\ x \neq y \right\}$$

is bounded. In this case the function f is uniformly continuous, so it has a unique continuous extension to $\overline{U}$. Observe that the 1-Hölder functions are the Lipschitz functions and it is important to note that these functions are also α-Hölder for any $\alpha \in (0,1]$.

If U is open, if all partial derivatives of f up to order $k \geq 0$ exist and if the partial derivatives of order k of f are α-Hölder, then f is said to be of class $C^{k,\alpha}$: *these functions are called multiply Hölder functions.* If $U' \subset U$ and $f|_{U'}$ is of class $C^{k,\alpha}$, we say that f is of class $C^{k,\alpha}$ on U' and write $f \in C^{k,\alpha}(U')$. Clearly the functions of class $C^{0,\alpha}$ are the α-Hölder functions and a function of class $C^{k,\alpha}$ is necessarily of class C^k.

In this work we show several properties about multiply Hölder functions. In Section 2 we introduce the special class of subconvex open sets, which are specially adapted to the study of $C^{k,\alpha}$ functions (see [6, 1]). In Section 3, given a subconvex open set U, we define the inner distance to a boundary point of U. The main result of the section — Theorem 3.1 — provides a condition for a differentiable function on U to be α-Hölder continuous. In Section 4 we prove some general properties of $C^{k,\alpha}$ functions. Among other basic facts, we study the Hölder classes of compositions and inverses of $C^{k,\alpha}$ functions. Finally, in Section 5, we study some extension properties of $C^{k,\alpha}$ functions. For example, Theorem 5.2 shows that a $C^{k,\alpha}$ function defined around a C^1 manifold of codimension ≥ 2 always extends to the manifold.

2. Subconvex Sets

Let $U \subset \mathbb{R}^n$ be an open connected set. Given $x, y \in U$, we define $\mathrm{dist}_U(x, y)$ as the infimum of the lengths of the rectifiable curves in U connecting x with y. The function

$$\mathrm{dist}_U : U \times U \to [0, +\infty)$$

defines a metric called the inner metric of U. This metric is the intrinsic metric induced by the euclidean metric in U — for more details see [5]. The set U will be said subconvex if the inner metric dist_U is strongly equivalent to the euclidean metric: this is equivalent to the existence of a constant $d > 0$ such that

$$\mathrm{dist}_U(x, y) < d|x - y|, \quad x, y \in U.$$

In this case we say that U is subconvex of constant $d > 0$.

The following proposition establish some basic facts about subconvex sets. In particular, the proposition gives several examples of subconvex sets that are not convex sets.

Proposition 2.1.

1. *A convex set is subconvex.*
2. *If $U \subset \mathbb{R}^m$ is subconvex and $M \subset U$ is a smooth manifold of codimension ≥ 2, then $U \backslash M$ is subconvex.*
3. *If $U \subset \mathbb{R}^m$ is open, bounded and connected, and its boundary ∂U is a smooth hypersurface, then U is subconvex.*
4. *Let $V \subset \mathbb{R}^m$ be open and $K \subset V$ be compact and connected. Then there exists $U \subset V$ open, bounded and subconvex such that $K \subset U$.*

5. *If $U \subset \mathbb{R}^m$ is subconvex and $f\colon U \to \mathbb{R}^n$ is bilipschitz, then $f(U)$ is subconvex.*

6. *Let $U \subset \mathbb{R}^m$ be subconvex and let $f\colon U \to \mathbb{R}^n$ be differentiable such that $|df|$ is bounded on U. Then f is Lipschitz.*

Proof. The first assertion is obvious.

Let $U \subset \mathbb{R}^m$ be subconvex of constant d and let $M \subset U$ be a smooth manifold of codimension ≥ 2. Let $x, y \in U$ be distinct. Then

$$\mathrm{dist}_U(x, y) < d|x - y|$$

and, from the definition of $\mathrm{dist}_U(x, y)$, we can find a curve γ in U connecting x with y, such that

$$\ell(\gamma) < d|x - y|.$$

Since M has codimension ≥ 2, the curve γ can be deformed into a curve γ' in U avoiding the manifold M and being close to γ such that

$$\ell(\gamma') < d|x - y|.$$

Therefore assertion (2) is proved.

Let $U \subset \mathbb{R}^m$ be open, bounded and connected, and suppose that ∂U is a smooth hypersurface. If S is a connected component of ∂U, let

$$d_S\colon S \times S \to [0, +\infty)$$

be the geodesic metric in S. Since S is a compact hypersurface, the geodesic metric d_S is equivalent to the euclidean metric, so there exists $c \geq 1$ such that

$$d_S(x, y) \leq c|x - y|, \quad x, y \in S. \tag{2.1}$$

Since ∂U has finitely many connected components, we can assume that the constant c is independent of the component S. Let $x, y \in U$ be distinct. Deforming the euclidean segment from x to y we find a smooth simple curve γ connecting x with y, transverse to ∂U and such that

$$\ell(\gamma) < 2|x - y|. \tag{2.2}$$

Regarded as a set, γ has a total order induced by its orientation such that $x < y$. Thus, given $p, q \in \gamma$ with $p < q$, we denote by $[p, q]$ the compact segment of γ from p to q, and by (p, q) the open segment $[p, q]\backslash\{p, q\}$. Recall that each connected component of ∂U, since it is a compact hypersurface, separates the space $\mathbb{R}^m$ in two connected components. Therefore, for some $k \in \mathbb{N}$ we find points

$$x = p_1 < p_2 < \ldots < p_{2k} = y$$

in γ with the following properties:

- $(p_{2j-1}, p_{2j}) \subset U$, $j = 1, \ldots, k$.
- $(p_{2j}, p_{2j+1}) \subset \mathbb{R}^m\backslash U$, $j = 1, \ldots, k - 1$.
- For each $j = 1, \ldots, k - 1$, the points p_{2j} and p_{2j+1} belong to the same component of ∂U.

It follows from the last property and (2.1) that

$$d_S(p_{2j}, p_{2j+1}) \leq c|p_{2j} - p_{2j+1}|, \quad j = 1, \ldots, k - 1.$$

Thus if α_j is the minimal geodesic in ∂U connecting p_{2j} with p_{2j+1}, we have

$$\ell(\alpha_j) \leq c|p_{2j} - p_{2j+1}|, \quad j = 1, \ldots, k - 1.$$

Let γ' be the curve obtained from γ by replacing each $[p_{2j}, p_{2j+1}]$ by α_j.

Then

$$\begin{aligned}\ell(\gamma') &= \sum_{j=1}^{k} \ell([p_{2j-1}, p_{2j}]) + \sum_{j=1}^{k-1} \ell(\alpha_j)\\ &\le \sum_{j=1}^{k} \ell([p_{2j-1}, p_{2j}]) + \sum_{j=1}^{k-1} c|p_{2j} - p_{2j+1}|\\ &\le \sum_{j=1}^{k} \ell([p_{2j-1}, p_{2j}]) + c\sum_{j=1}^{k-1} \ell([p_{2j}, p_{2j+1}])\\ &\le c\ell(\gamma),\end{aligned}$$

so from (2.2),

$$\ell(\gamma') < 2c|x-y|.$$

Then we can deform $\gamma' \subset \overline{U}$ into a curve $\gamma'' \subset U$ connecting x with y, such that we also have

$$\ell(\gamma'') < 2c|x-y|,$$

which finishes the proof of assertion (3).

Assertion (4) is a consequence of assertion (3): we only have to take U being open, bounded, connected, with

$$K \subset U \subset V,$$

such that ∂U is a smooth hypersurface. So (4) holds.

Now, let $U \subset \mathbb{R}^m$ be subconvex and let $f\colon U \to \mathbb{R}^n$ be a bilipschitz map. Then there are constants $c_1, c_2 > 0$ such that

$$c_1|x-y| \le |f(x) - f(y)| \le c_2|x-y|, \quad x, y \in U. \tag{2.3}$$

This guarantees that f is a homeomorphism, whence $f(U)$ is open and connected. Let $u, v \in f(U)$ be distinct. Since U is subconvex there exists $d > 0$ such that

$$\mathrm{dist}_U(x,y) < d|x-y|, \quad x, y \in U.$$

Then, since $f^{-1}(u), f^{-1}(v) \in U$, from the definition of dist_U we can find a rectifiable curve γ in U, connecting $f^{-1}(u)$ with $f^{-1}(v)$, such that

$$\ell(\gamma) < d \left| f^{-1}(u) - f^{-1}(v) \right| . \tag{2.4}$$

On the other hand, since f is Lipschitz of constant c_2, the curve $f(\gamma)$, which connects u with v, is such that

$$\ell(f(\gamma)) \leq c_2 \ell(\gamma).$$

From this, (2.4)and (2.3) we obtain

$$\begin{aligned} \ell(f(\gamma)) &\leq c_2 \ell(\gamma) \leq c_2 d \left| f^{-1}(u) - f^{-1}(v) \right| \\ &\leq c_2 d (1/c_1) |u - v|, \end{aligned}$$

which proves that $f(U)$ is subconvex.

Finally, assume the hypotheses of assertion (6). Then there exists a constant $C > 0$ such that

$$|df(x)| \leq C, \quad x \in U.$$

Given $x, y \in U$, since U is subconvex we can find a smooth curve $\gamma\colon [0,1] \to U$ with $\gamma(0) = x$ and $\gamma(1) = y$, such that

$$\ell(\gamma) < d|x - y|,$$

where $d > 0$ is a constant depending only on U. Then

$$\begin{aligned} |f(y) - f(x)| = \left| \int_0^1 (f(\gamma(t)))' \, dt \right| &\leq \int_0^1 |df(\gamma(t))| \, |\gamma'(t)| \, dt \\ &\leq C \int_0^1 |\gamma'(t)| \, dt = C\ell(\gamma) \\ &\leq Cd|x - y|, \end{aligned}$$

so assertion (6) is proved. □

3. Differentiability and the Hölder condition

Let $U \subset \mathbb{R}^n$ be a subconvex open set and consider a continuous function $f\colon U \to \mathbb{R}^n$. It is well known that, if f is differentiable, the Hölder property for f is strongly related with the geometry of ∂U and the behavior of f in relation with ∂U. In this section we prove that, if f extends continuously to a compact set $K \subset \partial U$, if $f|_K$ is α-Hölder and $df(x)$ is bounded in terms of the "inner" distance from x to K, then f α-Hölder.

Inner distance to the boundary

Let U be an open subconvex proper subset of $\mathbb{R}^m$. Given $x_0 \in U$ and $x_\infty \in \partial U$, we define the inner distance from x_0 to x_∞ as the infimum of the lengths of the continuous rectifiable curves

$$\gamma\colon [0,1) \to U, \quad \gamma(0) = x_0, \quad \lim_{t\to 1} \gamma(t) = x_\infty. \tag{3.1}$$

Let us see that this infimum is finite. Given

$$r > |x_0 - x_\infty|,$$

we can take a sequence of points $x_1, x_2, \ldots$ in U with $x_j \to x_\infty$ as $j \to \infty$, such that

$$|x_0 - x_1| + |x_1 - x_2| + |x_2 - x_3| + \ldots < r.$$

If U is subconvex of constant $d > 0$, for each $j \in \mathbb{N}$ we can take a continuous rectifiable curve γ_j connecting x_{j-1} with x_j, such that

$$\ell(\gamma_j) < d|x_{j-1} - x_j|.$$

Then the infinite juxtaposition

$$\gamma\colon = \gamma_1 * \gamma_2 * \cdots$$

defines a curve as in (3.1) such that

$$\ell(\gamma) < dr.$$

Therefore the infimum of the lengths of the curves in (3.1) is a finite number. We keep the notation $\mathrm{dist}_U(x_0, x_\infty)$ for the inner distance between $x_0 \in U$ and $x_\infty \in \partial U$. We note that, since the number $r > |x_0 - x_\infty|$ above is arbitrarily chosen, we have

$$\mathrm{dist}_U(x_0, x_\infty) \leq d|x_0 - x_\infty|.$$

Now, let $K \subset \partial U$ be nonempty. Given $x \in U$, we define the inner distance from x to K as

$$\delta(x) \colon = \inf\{\mathrm{dist}_U(x, x_\infty) \colon x_\infty \in K\}.$$

Theorem 3.1. *Let U be an open subconvex proper subset of $\mathbb{R}^m$ and consider $K \subset \partial U$ nonempty. Let $f \colon U \cup K \to \mathbb{R}^n$ be continuous, such that $f|_K$ is α-Hölder for some $\alpha \in (0, 1]$. Given $x \in U$, let $\delta(x)$ denote the inner distance in U from x to K. Suppose that f is differentiable in U and such that, for some $c > 0$,*

$$|df(x)| \leq c\delta(x)^{\alpha-1}, \quad x \in U. \tag{3.2}$$

Then f is α-Hölder.

We begin with an elementary lemma.

Lemma 3.2. *Let $g \colon [0, 1] \to \mathbb{R}^n$ be continuous and such that, for some $c > 0$ and $\alpha \in (0, 1]$, we have $|g'(t)| \leq ct^{\alpha-1}$ for all $t \in (0, 1)$. Then*

$$|g(u) - g(v)| \leq \frac{c}{\alpha}|u - v|^\alpha, \quad u, v \in [0, 1].$$

Proof. Suppose that $1 > u \geq v > 0$. Given $n \in \mathbb{N}$, let $v = t_0 < t_1 < \ldots < t_n = u$ be the regular partition of norm $(u - v)/n$. By the Mean Value Theorem, there exists numbers $s_j \in [t_{j-1}, t_j]$ such that $g(t_j) - g(t_{j-1}) = g'(s_j)(u-v)/n$. Then

$$\begin{aligned} |g(u) - g(v)| \leq \sum_{j=1}^{n} |g(t_j) - g(t_{j-1})| &\leq \sum_{j=1}^{n} |g'(s_j)|(u-v)/n \\ &\leq \sum_{j=1}^{n} c s_j^{\alpha-1}(u-v)/n. \end{aligned}$$

Thus, if $n \to \infty$, we obtains that

$$|g(u) - g(v)| \leq \int_{v}^{u} ct^{\alpha-1} dt = \frac{c}{\alpha}\left(u^{\alpha} - v^{\alpha}\right) \leq \frac{c}{\alpha}(u-v)^{\alpha},$$

so the result easily follows. □

Proof of Theorem 3.1. It is enough to show that $f|_U$ is α-Hölder. Let $x, y \in U$ be arbitrary. Suppose that U is subconvex of constant $d > 0$.

Case 1: Suppose that $\delta(x) < (d+1)|x-y|$ and $\delta(y) < (d+1)|x-y|$. Since

$$\delta(x) < (d+1)|x-y|,$$

there exist $x_\infty \in K$ and a smooth rectifiable curve

$$\gamma\colon (0,1] \to U, \quad \gamma(1) = x, \quad \lim_{t\to 0} \gamma(t) = x_\infty$$

such that

$$\ell(\gamma) < (d+1)|x-y|.$$

Clearly we can continuously extend γ to $[0,1]$ defining $\gamma(0) = x_\infty$. Moreover we can assume that γ has constant velocity, that is

$$|\gamma'(t)| = \ell(\gamma), \quad t \in (0,1].$$

Define the function

$$g(t) = f(\gamma(t)), \quad t \in [0,1].$$

Then, if $t \in (0, 1]$,

$$|g'(t)| = |df(\gamma(t))||\gamma'(t)| \leq c[\delta(\gamma(t))]^{\alpha-1}\ell(\gamma). \tag{3.3}$$

Since the curve $\gamma|_{(0,t]}$ connects $x_\infty \in K$ with $\gamma(t) \in U$, we have

$$\delta(\gamma(t)) \leq \ell\left(\gamma|_{(0,t]}\right) = t\ell(\gamma),$$

so from (3.3) we obtain

$$|g'(t)| \leq c\ell(\gamma)^\alpha t^{\alpha-1}. \tag{3.4}$$

Then it follows from Lemma 3.2 that

$$|g(u) - g(v)| \leq \frac{c}{\alpha}\ell(\gamma)^\alpha |u - v|^\alpha, \quad u, v \in [0, 1].$$

In particular,

$$|f(x) - f(x_\infty)| = |g(1) - g(0)| \leq \frac{c}{\alpha}\ell(\gamma)^\alpha$$

and so, since $\ell(\gamma) < (d + 1)|x - y|$,

$$|f(x) - f(x_\infty)| \leq (d + 1)^\alpha \frac{c}{\alpha}|x - y|^\alpha.$$

Analogously, we find $y_\infty \in K$ such that

$$|f(y) - f(y_\infty)| \leq (d + 1)^\alpha \frac{c}{\alpha}|x - y|^\alpha.$$

On the other hand, since $f|_{\overline{K}}$ is α–Hölder, there exists a constant $c_\infty > 0$ depending only on f such that

$$|f(x_\infty) - f(y_\infty)| \leq c_\infty |x_\infty - y_\infty|^\alpha.$$

Using the last three inequalities together we finally obtain

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f(x_\infty)| + |f(y) - f(y_\infty)| + |f(x_\infty) - f(y_\infty)| \\
&\leq 2(d + 1)^\alpha \frac{c}{\alpha}|x - y|^\alpha + c_\infty |x_\infty - y_\infty|^\alpha \\
&\leq 2(d + 1)^\alpha \frac{c}{\alpha}|x - y|^\alpha + c_\infty \Big(\delta(x) + |x - y| + \delta(y)\Big)^\alpha \\
&\leq 2(d + 1)^\alpha \frac{c}{\alpha}|x - y|^\alpha + c_\infty \Big((2d + 3)|x - y|\Big)^\alpha \\
&\leq \left(2(d + 1)^\alpha \frac{c}{\alpha} + c_\infty (2d + 3)^\alpha\right)|x - y|^\alpha.
\end{aligned}$$

Case 2: If the first case does not happen, without loss of generality we can assume that

$$\delta(x) \leq \delta(y) \quad \text{and} \quad (d+1)|x-y| \leq \delta(y).$$

Then, since U is subconvex of constant d, we find a smooth curve

$$\gamma\colon [0,1] \to U, \quad \gamma(0) = x, \quad \gamma(1) = y,$$

such that

$$\ell(\gamma) < d|x-y|.$$

We assume that γ has constant velocity:

$$|\gamma'(t)| = \ell(\gamma), \quad t \in [0,1].$$

From the definition of inner distance we see that, for any $t \in [0,1]$,

$$\delta(y) \leq \mathrm{dist}_U(y, \gamma(t)) + \delta(\gamma(t)).$$

Thus, since $\mathrm{dist}_U(y, \gamma(t)) \leq \ell(\gamma)$,

$$\delta(y) \leq \ell(\gamma) + \delta(\gamma(t)),$$

whence

$$\begin{aligned}
\delta(\gamma(t)) &\geq \delta(y) - \ell(\gamma) \geq (d+1)|x-y| - \ell(\gamma) \\
&\geq (d+1)|x-y| - d|x-y| \\
&\geq |x-y|.
\end{aligned}$$

Therefore

$$\begin{aligned}
|f(y) - f(x)| &= \left| \int_0^1 df\big(\gamma(t)\big)\gamma'(t)dt \right| \leq \int_0^1 c\,[\delta(\gamma(t))]^{\alpha-1}\,|\gamma'(t)|dt \\
&\leq \int_0^1 c|x-y|^{\alpha-1}\ell(\gamma)dt \leq \int_0^1 cd|x-y|^{\alpha}dt \\
&\leq cd|x-y|^{\alpha}.
\end{aligned}$$

□

4. Some general properties

In this section we prove some general properties of $C^{k,\alpha}$ functions. For example, Theorem 4.2 gives the Hölder class of a composition of multiply Hölder functions, and Theorem 4.3 gives conditions for the inverse of a $C^{k,\alpha}$ function to be a $C^{k,\alpha}$ function. These theorems complement some known similar results; see Proposition 1.2.7 and Theorem 1.3.4 in [3]. We start with the following property summarizing some elementary properties of multiply Hölder functions. Although these properties can be founded in [3], we include a complete proof here for the convenience of the reader.

Theorem 4.1. *Let $U \subset \mathbb{R}^m$ be open, bounded and subconvex, and consider $f\colon U \to \mathbb{R}^n$, $g\colon U \to \mathbb{R}^p$, $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0,1]$.*

1. *If $f \in C^{k,\alpha}(U)$, then f is bounded.*
2. *If $f \in C^{k+1,\alpha}(U)$, then $f \in C^{k,1}(U)$ — in particular $f \in C^{k,\alpha}(U)$.*
3. *If $f \in C^{k,\alpha}(U)$, then f extends to $\overline{U}$ as an α-Hölder function.*
4. *If $n = p$ and $f, g \in C^{k,\alpha}(U)$, then $f \pm g \in C^{k,\alpha}(U)$.*
5. *If $H\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$ is bilinear and $f, g \in C^{k,\alpha}(U)$, then $H(f,g)$ is of class $C^{k,\alpha}$.*
6. *If f extends as a C^{k+1} function on a neighborhood of $\overline{U}$, then $f \in C^{k,\alpha}(U)$.*

Proof. Suppose that $f \in C^{0,\alpha}(U)$. Then there exists $c > 0$ such that

$$|f(y) - f(x)| \leq c|y - x|^\alpha, \quad x, y \in U.$$

Fix $a \in U$ and let $x \in U$ be arbitrary. Then, since $|f(x) - f(a)| \leq c|x - a|^\alpha$,

$$|f(x)| \leq |f(a)| + c|x - a|^\alpha \leq |f(a)| + c(\mathrm{Diam}U)^\alpha,$$

which proves item (1) for $k = 0$. Then, item (1) will be proved if we prove item (2): if $f \in C^{k,\alpha}(U)$, by successively applications of item (2) we have $f \in C^{0,\alpha}(U)$, whence — by item (1) for $k = 0$ — the function f is bounded. Let us prove item (2). Suppose that $f \in C^{1,\alpha}(U)$. Then any partial derivative $\frac{\partial f}{\partial x_i}$ belongs to $C^{0,\alpha}(U)$. Thus, by item (1) for $k = 0$ the functions $\frac{\partial f}{\partial x_i}$ are bounded. Then it follows from (6) of Proposition 2.1 that f is Lipschitz, which proves item (2) for $k = 0$. Suppose now that $f \in C^{k+1,\alpha}(U)$. Then any partial derivative of order k of f belongs to $C^{1,\alpha}(U)$. So, by item (2) for $k = 0$, any partial derivative of order k of f belongs to $C^{0,1}(U)$, which means that $f \in C^{k,1}(U)$; item (2) is proved.

If $f \in C^{k,\alpha}(U)$, it follows from item (2) that $f \in C^{0,\alpha}(U)$, so f extends to $\overline{U}$ as an α-Hölder function and item (3) is proved.

It suffices to prove item (4) for the sum of functions; the other case is similar. Suppose that $f, g \in C^{0,\alpha}(U)$. Then we can find $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^{\alpha}, \quad |g(x) - g(y)| \leq c|x - y|^{\alpha}, \quad x, y \in U.$$

Therefore

$$\begin{aligned} |(f(x) + g(x)) - (f(y) + g(y))| &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq 2c|x - y|^{\alpha}, \end{aligned}$$

which proves item (4) for $k = 0$. Suppose that item (4) is true for $k = l$, and let $f, g \in C^{l+1,\alpha}(U)$. Then the partial derivatives

$$\frac{\partial f}{\partial x_i}, \quad \frac{\partial g}{\partial x_i}, \quad i = 1, \ldots, m$$

belong to $\in C^{l,\alpha}(U)$. Thus, by the inductive hypothesis, the partial derivatives

$$\frac{\partial (f + g)}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}, \quad i = 1, \ldots, m$$

belong to $C^{l,\alpha}(U)$. Therefore $f + g \in C^{l+1,\alpha}(U)$, so item (4) is proved.

Now, assume the hypotheses of item (5). Since H is bilinear there is a constant $c_H > 0$ such that

$$|H(u,v)| \leq c_H |u||v|, \quad u \in \mathbb{R}^n, v \in \mathbb{R}^p.$$

Suppose first that $k = 0$. Then there exists $c > 0$ such that

$$|f(x) - f(y)| \leq c|x-y|^\alpha, \quad |g(x) - g(y)| \leq c|x-y|^\alpha, \quad x, y \in U.$$

Moreover, by item (1) there exist $C > 0$ such that

$$|f(x)| \leq C, \quad |g(x)| \leq C, \quad x \in U.$$

Thus, if $x, y \in U$,

$$\begin{aligned}
|H(f(x), g(x)) &- H(f(y), g(y))| \\
&\leq |H(f(x), g(x)) - H(f(x), g(y))| \\
&\qquad\qquad + |H(f(x), g(y)) - H(f(y), g(y))| \\
&\leq |H(f(x), g(x) - g(y))| + |H(f(x) - f(y), g(y))| \\
&\leq c_H |f(x)||g(x) - g(y)| + c_H |f(x) - f(y)||g(y)| \\
&\leq c_H C \left(c|x-y|^\alpha\right) + c_H \left(c|x-y|^\alpha\right) C \\
&\leq 2c_H C c |x-y|^\alpha,
\end{aligned}$$

which proves item (5) for $k = 0$. Suppose item (5) is true for $k = l$, and let $f, g \in C^{l+1,\alpha}(U)$. Since $f, g \in C^{l+1,\alpha}(U)$, for any $i \in \{1, \ldots, m\}$ we have $\frac{\partial f}{x_i}, \frac{\partial g}{\partial x_i} \in C^{l,\alpha}(U)$ and — by item (2) — we also have $f, g \in C^{l,\alpha}(U)$. Then, by the inductive hypothesis, $H\left(\frac{\partial f}{x_i}, g\right)$ and $H\left(f, \frac{\partial g}{\partial x_i}\right)$ belong to $C^{l,\alpha}(U)$, whence — by item (4) — the partial derivative

$$\frac{\partial H(f,g)}{\partial x_i} = H\left(\frac{\partial f}{x_i}, g\right) + H\left(f, \frac{\partial g}{\partial x_i}\right)$$

belongs to $C^{l+1,\alpha}(U)$. Therefore item (5) is proved.

Finally, assume that f extends as a C^{k+1} function on a neighborhood of $\overline{U}$. Let $\mathfrak{f}$ be a partial derivative of order k of f. Then, since f extends in the class C^{k+1} to a neighborhood of the compact set $\overline{U}$, each partial derivative of $\mathfrak{f}$ extends continuously to that neighborhood of $\overline{U}$, so the partial derivatives of $\mathfrak{f}$ are bounded on U. It follows from (6) of Proposition 2.1 that $\mathfrak{f}$ is Lipschitz. That is, $\mathfrak{f} \in C^{0,\alpha}(U)$, which means that $f \in C^{k,\alpha}(U)$. □

Theorem 4.2. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. Let $f\colon U \to V$ be of class $C^{k,\alpha}$ and $g\colon V \to \mathbb{R}^p$ of class $C^{k,\beta}$, where $k \in \mathbb{Z}_{\geq 0}$ and $\alpha, \beta \in (0,1]$. Then, if U is bounded and subconvex, $g \circ f$ is of class $C^{k,\alpha\beta}$.*

Proof. Suppose first that $f \in C^{0,\alpha}(U)$ and $g \in C^{0,\beta}(V)$. Thus there are constants $c_f, c_g > 0$ such that

$$|f(x) - f(y)| \leq c_f |x-y|^\alpha, \quad x, y \in U$$

and

$$|g(x) - g(y)| \leq c_g |x-y|^\beta, \quad x, y \in V.$$

Then, if $x, y \in U$,

$$\begin{aligned} |g \circ f(x) - g \circ f(y)| &\leq c_g |f(x) - f(y)|^\beta \leq c_g \left(c_f |x-y|^\alpha\right)^\beta \\ &\leq c_g c_f^\beta |x-y|^{\alpha\beta}, \end{aligned}$$

which proves the proposition for $k = 0$. Suppose as inductive hypothesis that the proposition holds for some $k \in \mathbb{Z}_{\geq 0}$. Let $f \in C^{k+1,\alpha}(U)$ and $g \in C^{k+1,\beta}(V)$. Then $df \in C^{k,\alpha}(U)$ and $dg \in C^{k,\beta}(V)$ and, by (2) of Proposition 4.1, we also have $f \in C^{k,\alpha}(U)$. It follows from the inductive hypothesis that $dg(f) \in C^{k,\alpha\beta}(U)$. Then, since

$$df \in C^{k,\alpha}(U) \subset C^{k,\alpha\beta}(U),$$

from (5) of Proposition 4.1 we see that

$$d(g \circ f) = dg(f) \cdot df$$

belongs to $C^{k,\alpha\beta}(U)$, which means that $g \circ f \in C^{k+1,\alpha\beta}(U)$. □

Corollary 4.3. *Let $U \subset \mathbb{R}^m$ be open, bounded and subconvex, and let $f\colon U \to \mathbb{R}$ be of class $C^{k,\alpha}$, where $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0,1]$. If $|f| > \epsilon$ for some $\epsilon > 0$, then $1/f$ is of class $C^{k,\alpha}$.*

Proof. Since $|f| > \epsilon$ and $f(U)$ is connected, without loss of generality we can assume that

$$f(U) \subset (\epsilon, +\infty).$$

Furthermore, from (1) of Proposition 4.1 we find $c > 0$ such that

$$f(U) \subset V\colon = (\epsilon, c).$$

Since (6) of Proposition 4.1 guarantees that $g\colon V \to \mathbb{R}$ defined by $g(x) = 1/x$ is of class $C^{k,1}$, the corollary follows from Proposition 4.2 □

Theorem 4.4. *Let $U \subset \mathbb{R}^m$ be a bounded subconvex open set and let $f\colon U \to \mathbb{R}^m$ be of class $C^{k,\alpha}$ for $k \geq 1$ and $\alpha \in (0,1]$. From Proposition 4.1 we see that $f\colon U \to \mathbb{R}^m$ and $df\colon U \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ extend continuously to $\overline{U}$. Suppose that the extension of f to $\overline{U}$ is univalent and the extension of df to $\overline{U}$ takes values in $GL(m, \mathbb{R})$. Then $f(U)$ is a bounded subconvex open set and $f^{-1}\colon f(U) \to U$ is of class $C^{k,\alpha}$.*

Proof. We still denote by f and df the extensions of f and df to $\overline{U}$. Since $k \geq 1$, from (1) and (2) of Proposition 4.1 the partial derivatives of f are bounded. Then, it follows from (6) of Proposition 2.1 that f is Lipschitz. Let us prove that $f^{-1}\colon f(U) \to U$ is also Lipschitz. Otherwise, there are points $x_n, y_n \in U$, $n \in \mathbb{N}$ such that

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.1}$$

Since U is bounded we can assume that, for some $a, b \in \overline{U}$,

$$x_n \to a, \quad y_n \to b \quad \text{as} \quad n \to \infty.$$

Thus, if $a \neq b$ we $f(a) \neq f(b)$ and therefore

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} \to \frac{|f(a) - f(b)|}{|a - b|} \neq 0, \tag{4.2}$$

which contradicts (4.1). So we assume $a = b$. Since $df(a) \in \mathrm{GL}(n, \mathbb{R})$, there exists a constant $c_a > 0$ such that

$$|df(a) \cdot v| \geq 2c_a|v|, \quad v \in \mathbb{R}^n \backslash \{0\}.$$

On the other hand, suppose that U is subconvex of constant d. Then, since df is continuous on $\overline{U}$, there is a neighborhood Ω of a in $\overline{U}$ such that

$$|df(x) - df(a)| \leq c_a/d, \quad x \in \Omega. \tag{4.3}$$

Since U is subconvex of constant d, we can find a smooth curve $\gamma\colon [0,1] \to U$ such that $\gamma(0) = x_n$, $\gamma(1) = y_n$ and

$$\ell(\gamma) < d|x_n - y_n|.$$

We can assume x_n and y_n to be so close to a such that the curve γ is contained in Ω. Then it follows from (4.3) that

$$|df(\gamma(t)) - df(a)| \leq c_a/d, \quad t \in [0,1]. \tag{4.4}$$

Thus, if we define

$$g(t) = f(\gamma(t)), \quad t \in [0,1],$$

we can write

$$g(t) = df(a) \cdot \gamma(t) + \delta(t),$$

where we have

$$\begin{aligned} |\delta'(t)| &= |df(\gamma(t)) \cdot \gamma'(t) - df(a) \cdot \gamma'(t)| \\ &= |\,[df(\gamma(t)) - df(a)] \cdot \gamma'(t)| \\ &\leq |df(\gamma(t)) - df(a)| \cdot |\gamma'(t)| \\ &\leq (c_a/d)|\gamma'(t)|. \end{aligned}$$

Therefore

$$\begin{aligned}
|f(y_n)-f(x_n)| &= \left|\int_0^1 g'(t)dt\right| = \left|\int_0^1 (df(a)\cdot\gamma'(t)+\delta'(t))dt\right| \\
&\geq \left|\int_0^1 df(a)\cdot\gamma'(t)dt\right| - \left|\int_0^1 \delta'(t)dt\right| \\
&\geq \Big|df(a)\cdot(y_n-x_n)\Big| - (c_a/d)\int_0^1 |\gamma'(t)|dt \\
&\geq 2c_a|y_n-x_n| - (c_a/d)\ell(\gamma) \\
&\geq 2c_a|y_n-x_n| - (c_a/d)(d|y_n-x_n|) \\
&\geq c_a|y_n-x_n|,
\end{aligned}$$

whence we conclude that

$$\frac{|f(y_n)-f(x_n)|}{|y_n-x_n|} \geq c_a$$

for n large enough, which contradicts (4.1).

Now, the fact of f being bilipschitz implies that $f(U)$, like U, is open, bounded and subconvex.

Since df extends continuously to $\overline{U}$ and takes values in $\mathrm{GL}(n,\mathbb{R})$, the image of df is a compact connected set $K \subset \mathrm{GL}(n,\mathbb{R})$. Thus, from (4) of Proposition 2.1 we can find an open bounded subconvex set $\mathcal{U}$ in the space of $n\times n$ matrices, such that

$$K \subset \mathcal{U} \quad \text{and} \quad \overline{\mathcal{U}} \subset \mathrm{GL}(n,\mathbb{R}).$$

It follows from (6) of Proposition 4.1 that the function

$$\begin{aligned}
\mathcal{I}\colon \mathcal{U} &\to \mathrm{GL}(n,\mathbb{R}) \\
M &\mapsto M^{-1}
\end{aligned}$$

is of class $C^{l,1}$ for all $l \geq 0$. Since $k \geq 1$, the derivative of f^{-1} can be expressed as

$$df^{-1} = \mathcal{I} \circ df \circ f^{-1}. \tag{4.5}$$

Suppose first that $k = 1$. Then $df : U \to \mathrm{GL}(n, \mathbb{R})$ is of class $C^{0,\alpha}$. Thus, since f^{-1} is of class $C^{0,1}$, it follows from Theorem 4.2 that

$$df \circ f^{-1} : f(U) \to \mathcal{U}$$

is of class $C^{0,\alpha}$. Therefore, since $\mathcal{I} \in C^{0,1}$, it follows from (4.5) and Theorem 4.2 that df^{-1} is of class $C^{0,\alpha}$. Then f^{-1} is of class $C^{1,\alpha}$, so Theorem 4.4 holds true for $k = 1$. Suppose that Theorem 4.4 holds true for $k = l \geq 1$ and let f be satisfying the hypotheses of Theorem 4.4 for $k = l + 1$. Since f is of class $C^{l+1,\alpha}$, by (2) of Proposition 4.1 we have $f \in C^{l,1}(U)$ and therefore, by the inductive hypothesis, f^{-1} is also of class $C^{l,1}$. Then, since df is of class $C^{l,\alpha}$, it follows from Theorem 4.2 that $df \circ f^{-1}$ is of class $C^{l,\alpha}$. Therefore, since $\mathcal{I} \in C^{l,1}$, it follows from (4.5) and Theorem 4.2 that df^{-1} is of class $C^{l,\alpha}$, which means that f^{-1} is of class $C^{l+1,\alpha}$. □

5. Extension properties of multiply Hölder functions

In this section we prove a couple of results about the extension properties of $C^{k,\alpha}$ functions.

Proposition 5.1. *Let $U \subset \mathbb{R}^m$ be a subconvex open set, let $U_1, U_2 \subset U$ be such that $U \subset \overline{U_1} \cup \overline{U_2}$, let $f : U \to \mathbb{R}^n$ be continuous, and consider $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0, 1]$. Then the following properties hold:*

1. *If $f|_{U_1}$ and $f|_{U_2}$ are α-Hölder, then f is α-Hölder.*
2. *If $f \in C^k$, if the sets U_1 and U_2 are open, and if $f|_{U_1}$ and $f|_{U_2}$ are of class $C^{k,\alpha}$, then f is of class $C^{k,\alpha}$.*

Proof. Suppose that $f|_{U_1}$ and $f|_{U_2}$ are α-Hölder. Then we find $c > 0$ such that

$$|f(x) - f(y)| \leq c|x-y|^\alpha \quad \text{whenever} \quad x, y \in \overline{U_1} \text{ or } x, y \in \overline{U_2}. \tag{5.1}$$

Then, since $U \subset \overline{U_1} \cup \overline{U_2}$, it suffices to find an inequality like above when $x \in \overline{U_1}$, $y \in \overline{U_2}$ and when $x \in \overline{U_2}$, $y \in \overline{U_1}$. We assume $x \in \overline{U_1}$, $y \in \overline{U_2}$ — the other case is similar. Since U is subconvex, we find a smooth curve γ in U connecting x with y such that

$$\ell(\gamma) < d|x-y|, \tag{5.2}$$

where the constant $d > 0$ depends only on U. Since γ is connected, $\gamma \subset \overline{U_1} \cup \overline{U_2}$ and γ meets both sets $\overline{U_1}$ and $\overline{U_2}$, we can find $z \in \gamma$ such that

$$z \in \overline{U_1} \cap \overline{U_2}.$$

Thus, from (5.1) we have

$$|f(x) - f(z)| \leq c|x-z|^\alpha \quad \text{and} \quad |f(z) - f(y)| \leq c|z-y|^\alpha,$$

whence

$$|f(x) - f(y)| \leq c|x-z|^\alpha + c|z-y|^\alpha \leq c\ell(\gamma)^\alpha + c\ell(\gamma)^\alpha \leq 2c\ell(\gamma)^\alpha,$$

and from (5.2),

$$|f(x) - f(y)| \leq 2cd^\alpha |x-y|^\alpha.$$

Assertion (1) is proved.

Suppose now that U_1 and U_2 are open, $f \in C^k$ and $f|_{U_1}$ and $f|_{U_2}$ are of class $C^{k,\alpha}$. Let g be any partial derivative of order k of f. Since $f|_{U_1}$ and $f|_{U_2}$ are of class $C^{k,\alpha}$, we have that $g|_{U_1}$ and $g|_{U_2}$ are α-Hölder. Then, since $f \in C^k$ means that g is continuous, it follows from assertion (1) that g is α-Hölder, which proves assertion (2). □

Theorem 5.2. *Let U be an open subset of $\mathbb{R}^m$ and let $M \subset U$ be a proper embedded C^1 manifold of codimension ≥ 2. Let $f\colon U \backslash M \to \mathbb{R}^n$ be of class $C^{k,\alpha}$ for some $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0,1]$. Then f extends to U in the class $C^{k,\alpha}$.*

We need the following lemma.

Lemma 5.3. *Let U be an open subset of $\mathbb{R}^m$ and let $M \subset U$ be a proper embedded C^1 manifold of codimension ≥ 1. Let $f\colon U\backslash M \to \mathbb{R}^n$ be of class C^1. Suppose that f and its partial derivatives extend continuously to U. Then f extends to U in the class C^1.*

Proof. Let $\bar{f}\colon U \to \mathbb{R}^n$ be the extension of f to U. In view of the hypotheses, it is enough to prove that, given $p \in M$, we can find a C^1 coordinate system $(x_1, \ldots, x_m)$ around p such that the partial derivatives of $\bar{f}$ at p exist and we have

$$\frac{\partial \bar{f}}{\partial x_j}(p) = \lim_{x \to p} \frac{\partial f}{\partial x_j}(x), \quad j = 1, \ldots, m. \tag{5.3}$$

Fix $p \in M$. Consider affine coordinates such that the canonical unitary vectors $e_1, \ldots, e_m$ are transverse to M at p, and fix $j \in \{1, \ldots, m\}$. Since e_j is transverse to M at p, for $t \in \mathbb{R}^*$ small enough the euclidean segment $[p, p + te_j]$ intersects M only at p. Thus, if we set $f = (f_1, \ldots, f_m)$ and $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_m)$, by the Mean Value Theorem we have

$$\begin{aligned} \frac{\bar{f}(p+te_j) - \bar{f}(p)}{t} &= \left(\frac{\bar{f}_1(p+te_j) - \bar{f}_1(p)}{t}, \ldots, \frac{\bar{f}_m(p+te_j) - \bar{f}_m(p)}{t} \right) \\ &= \left(\frac{\partial f_1}{\partial x_j}(w_1), \ldots, \frac{\partial f_m}{\partial x_j}(w_m) \right), \end{aligned}$$

where $w_1, \ldots, w_m$ are points in the open segment

$$(p, p + te_j) \subset U\backslash M.$$

Then, since the points $w_1, \ldots, w_m$ tend to p as t tends to 0, we have

$$\begin{aligned} \lim_{t \to 0} \frac{\bar{f}(p+te_j) - \bar{f}(p)}{t} &= \left(\lim_{x \to p} \frac{\partial f_1}{\partial x_j}(x), \ldots, \lim_{x \to p} \frac{\partial f_m}{\partial x_j}(x) \right) \\ &= \lim_{x \to p} \frac{\partial f}{\partial x_j}(x). \end{aligned}$$

□

Proof of Theorem 5.2. Suppose first that $k = 0$. Then $f\colon U \setminus M \to \mathbb{R}^n$ is α-Hölder and therefore f extends as an α-Hölder function to

$$\overline{U\backslash M} \supset U,$$

which proves the proposition for $k = 0$. Suppose now that the proposition holds true for $k = l \in \mathbb{Z}_{\geq 0}$ and let $f\colon U\backslash M \to \mathbb{R}^n$ be of class $C^{l+1,\alpha}$. Our intention is to apply Lemma 5.3 to f. Thus, let us show that f and its partial derivatives extend continuously to U. To do so it is enough to show that each $p \in M$ has a neighborhood Ω in U such that the restrictions of f and its derivatives to $\Omega\backslash M$ are uniformly continuous. Fix $p \in M$ and let $\Omega \subset U$ be an open ball centered at p. Since Ω is convex, by (1) and (2) of Proposition 2.1 we have that $\Omega\backslash M$ is subconvex. Thus, since f is of class $C^{l+1,\alpha}$, it follows from (2) of Proposition 4.1 that f and its derivatives are of class $C^{0,\alpha}$ on $\Omega\backslash M$, so they are uniformly continuous on $\Omega\backslash M$. Therefore, by Lemma 5.3, the function f is the restriction to $U\backslash M$ of a C^1 function $\bar{f}\colon U \to \mathbb{R}^n$. To complete the induction, we shall prove that the partial derivatives of $\bar{f}$ belong to $C^{l,\alpha}(U)$. Let g be a partial derivative of $\bar{f}$. Since $g|_{U\backslash M}$ is a partial derivative of f, we have that $g|_{U\backslash M}$ is of class $C^{l,\alpha}$. Therefore, by the inductive hypothesis, $g|_{U\backslash M}$ extends to U in the class $C^{l,\alpha}$, which means that g is of class $C^{l,\alpha}$. □

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Resumen

En este artículo mostramos varias propiedades de las funciones Hölder multiples. Estudiamos la clase Hölder de una composición de funciones Hölder multiples y demostramos que una función y su inversa pertenecen – bajo ciertas hipótesis – a la misma clase Hölder. También demostramos algunas propiedades de extensión de las funciones Hölder multiples; por ejemplo, demostramos que una función Hölder multiple siempre extiende, en la misma clase Hölder, a «conjuntos excepcionales» que son variedades de codimensión uno.

Palabras clave: Clases de Lipschitz (Hölder); Propiedades especiales de funciones de varias variables, condiciones de Hölder.

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EXISTENCE OF OBSERVABLE SETS FOR CONTROL SYSTEMS ON LIE GROUPS

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(*Presentado por R. Rabanal*)

Abstract

In this paper, we introduce the concept of observable sets for control systems. For this reason, we review general control systems on Lie groups with observation (output) functions and give the solution of the affine control system on a connected Lie group. Then, we study the existence of observable sets for linear control systems on examples.

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Keywords: *observable set; observability; affine control systems; linear control systems*

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1. Introduction

The main purpose of this paper is to introduce a new approach to observability by searching the existence of observable sets. Observability problem is one of the classical fundamental problem in control theory and it is important from the application points of view. We approach from differential geometric point of view.

In 1972, the utility of Lie groups and coset spaces in modelling system dynamics has been first highlighted by Brokett, through group manifolds, [6]. Later, in 1977, observability of nonlinear control systems has been studied from differential geometric point of views by Hermann and Krener, [8]. In 1990, [7] by Cheng, Dayawansa, and Martin, is the first key paper directly addressing observability on Lie groups and the authors have studied local and global observabilities using Lie algebraic methods, leveraging the structure of Lie groups and coset spaces.

For a linear control system on $\mathbb{R}^n$, $\Sigma = (\mathbb{R}^n, D, h, \mathbb{R}^s)$ is given by the following data:

$$\dot{x} = Ax + Bu$$
$$h(x) = Cx,$$

where $x \in \mathbb{R}^n$, A, B and C are matrices of appropriate orders and $h : \mathbb{R}^n \to \mathbb{R}^s$ is a linear function. The well-known observability rank condition by Kalman is

$$\Sigma \textit{ is observable} \iff \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n.$$

In general, observability of control systems on Lie groups has been studied on the characterization of the set of indistinguishable elements of the state space in order to find conditions for local and global observabilities, [3], [4] and [5]. In our work, we focus on the complements of this kind of sets of indistinguishable elements of the state spaces, which we will call them observable sets.

Our work consists of five sections. In the second section, we review shortly control systems on Lie groups with observation. In the third section, we introduce observable sets by giving some definitions on observability. Moreover, we give the solution for the affine control system on a Lie group G with some output function and the observability is induced by the drift vector field of the system. In the fourth and fifth sections, we study the existence of observable sets for linear control systems both on Euclidean spaces and on Heisenberg group of dimension 3 on examples. Lastly, we write our conclusion of this work.

2. Control Systems on Lie groups with Observation

A control system $\Sigma = (G, \mathcal{D}, h, V)$ on a Lie group G with an observation (output) function is a four-tuple, where G and V are finite dimensional Lie groups, $\mathcal{D}$ is the dynamic and h is a smooth function between G and V. We assume that the state space G is connected. Here, V is called observation (output) space and h is called observation (output) function. The dynamic $\mathcal{D}$ is a set of smooth vector fields on G which are parametrized by the controls u and $u = (u_1, u_2, \ldots, u_k) \in \mathcal{U}$, where $\mathcal{U}$ is a family of piecewise constant real-valued functions.

Control system Σ induces a pseudo group of diffeomorphisms

$$G_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k} | Z^j \in \mathcal{D}, t_j \in \mathbb{R}, j \in \mathbb{N}\}$$

and a pseudo-semi group

$$S_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k} | Z^j \in \mathcal{D}, t_j \geq 0, j \in \mathbb{N}\}.$$

Besides,

$$G_\Sigma(g) = \{\varphi(g) | g \in G, \forall \varphi \in G_\Sigma\}$$

is the orbit, and

$$S_\Sigma(g) = \{\varphi(g) | g \in G, \forall \varphi \in S_\Sigma\}$$

is the positive orbit of the system at the state $g \in G$.

Denote by $L(G)$ the Lie algebra of G and consider $X, Y^1, \ldots, Y^d \in L(G)$ with right invariant directions. Besides, consider the elements $D, D^1, D^2, \ldots, , D^n$ of the derivation algebra $Der(L(G))$ of Lie group G. The derivation algebra $Der(L(G))$ is a Lie algebra consisting of endomorphisms D on $L(G)$ satisfying

$$D[X,Y] = [D(X), Y] + [X, D(Y)], \forall X, Y \in L(G).$$

Let χ be a linear vector field on G. Then, by [1], there is a derivation D associated to χ defined by

$$DY = -[\chi, Y], \forall Y \in L(G)$$

and the flow Φ_t of χ is related to D by

$$(d\Phi_t)_e = e^{tD}, \ for\ any\ t \in \mathbb{R}$$

and it follows that

$$\Phi_t(expY) = exp(e^{tD}Y)\ for\ any\ t \in \mathbb{R},\ Y \in L(G).$$

If D is an inner derivation, then $D = ad(X)$ for some $X \in L(G)$. This is a special case and it happens when the Lie group is semisimple.

Any linear vector field χ is an infinitesimal automorphism which means that its flow is a 1-parameter subgroup of automorphisms on G, [1], and $\chi + X$ forms an affine vector field on G, so are $(\chi^j + Y^j)$ for $j = 1, \ldots, d$. The dynamic of affine control system Σ on G is defined by the following differential equations together with an observation (output) function h:

$$\begin{cases} \dot{g}(t) = (\chi + X)(g(t)) + \sum_{j=1}^{d} (\chi^j + Y^j)(g(t))u(t) \\ \\ h(g) = v \in V. \end{cases} \tag{2.1}$$

Affine control systems represent a wide class of control systems, [9]. Indeed, if we consider, linear vector field χ is 0 in 2.1, then the system turns into an invariant control system, and if we consider, for each j, Y^j and X are 0 in 2.1, then the system turns into a bilinear control system. Moreover, if we consider this 2.1 system on an abelian Lie group, then it turns into a linear control system.

3. The Observable Set

In this section, we introduce the concept of observable sets. The idea of observable sets comes from the control sets presented in [2]. By this inspiration we search the existence of possible observable parts of a given system. Control sets are the maximal controllable parts of the systems and they are important to understand the controllability behaviour of the system. Similarly, observable sets can serve as a source for observing some behavior of a system. For this aim, we characterize observable sets and their properties which lead us to work on two examples in the next section.

We would like to review some of the fundamental definitions related to the observability of control systems. For the following three definitions, we consider a control system $\Sigma = (G, \mathcal{D}, h, V)$ on a connected Lie group G with an observation (output) function h which is a smooth function from G to V. Each vector field $Z^j \in \mathcal{D}$ parametrized by the control u (piecewise constant admissible control) induces a 1-parameter group of automorphisms (flow) $Z^j_{t_j}$, $t \in \mathbb{R}$, $j \in \mathbb{N}$, on G.

Definition 3.1 (Indistinguishable)**.** *Let g and l be two different elements of a Lie group G. Then these two elements are called indistinguishable, if the following equation holds*

$$h(Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k}(g)) = h(Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k}(l)), \forall t_j \geq 0, j \in \mathbb{N}.$$

In another words, g and l are called indistinguishable, if the following equation holds

$$h \circ S_\Sigma(g) = h \circ S_\Sigma(l).$$

We use $g \sim l$, when g and l are indistinguishable, and denote by I_g the set of all indistinguishable elements from g. We know that vector fields Z^j of the dynamic $\mathcal{D}$ are complete, [1], then the relationship $\sim$ is an equivalence relationship. Besides,

$$\tilde{g} = I_g = \{l \in G | g \sim l\} \iff I = \bigcup_g I_g,$$

where I is the set of all indistinguishable elements in G.

Definition 3.2. *For any given state g of a control system Σ, if g has a neighbourhood U such that all points in U are distinguishable from g, then the control system Σ is called locally observable at g. If the system Σ is locally observable at every $g \in G$, then it is called locally observable.*

Definition 3.3. *If, for all $l \in G$ different than g and $\forall t_j \geq 0, j \in \mathbb{N}$,*

$$h(Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k}(g)) \neq h(Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k}(l)),$$

then the control system Σ is called globally observable at g. If the control system Σ is globally observable at every $g \in G$, then it is called globally observable.

In another words, if there are no two different states forming the same curve in the observation (output) space, then it is said that the control system Σ is globally observable. Moreover, a system is globally observable if its internal state can be inferred from external outputs.

If we consider a linear control system $\Sigma = (\mathbb{R}^n, \mathcal{D}, h, \mathbb{R}^s)$ with an observation (output) function h which is a linear transformation, then local observability and global observability are the same and there is a well-known observability rank condition for this system. In general, for a control system $\Sigma = (G, \mathcal{D}, h, V)$ on a Lie group G, local observability and global observability are studied separately and global observability implies local observability.

In [3], the authors consider a linear control system Σ on a connected Lie group G with a Lie group homomorphism from the state space to the observation (output) space and give the solution for every admissible control. By using a similar approach, we have the following result.

Theorem 3.4. *Consider an affine control system* $\Sigma = (G, \mathcal{D}, h, V)$ *given by* (2.1)

$$\dot{g}(t) = (\chi + X)(g(t)) + \sum_{j=1}^{d} (\chi^j + Y^j)(g(t))u(t)$$

on a connected Lie group G *with the initial condition* $\gamma(0) = g$*, where* $g \in G$*,* $X, Y^1, \ldots, Y^d \in L(G)$*, the Lie algebra of* G*, and for* $j = 1, \ldots, d$*,* χ *and* χ^j *are linear vector fields on* G*, then the solution is*

$$\gamma(t) = (\chi + X)_t(\beta(t) \cdot g), t \in \mathbb{R},$$

where $(\chi+X)_t$ *is a one-parameter group of automorphisms on* G *induced by the drift vector field* $(\chi+X)$ *and* $\beta(t)$ *satisfies the following differential equation*

$$\dot{\beta(t)} = ((\chi + X)_{-t})_\star \circ \sum_{j=1}^{d} u_j(\chi^j + Y^j) \circ (\chi + X)(\beta(t)).$$

Proof. The dynamic $\mathcal{D}$ of the affine control system is

$$\mathcal{D} = \left\{ \begin{array}{ll} Z = (\chi + X) + u(\chi^j + Y^j) : & X, Y^j \in L(G)\ \forall,\ j = 1, \ldots d; \\ & \chi \text{ and } \chi^j \text{ are linear vector fields} \end{array} \right\}$$

If we differentiate $\gamma(t)$, then we have the following

$$\dot{\gamma(t)} = ((\chi + X)_t(\beta(t) \cdot g))_\star = ((\chi + X)(\beta(t) \cdot g))(\beta(t) \cdot g)_\star$$

$$= (\chi + X)(\beta(t) \cdot g)[(\chi + X)_{-t})_\star \circ \sum_{j=1}^{d} u_j(\chi^j + Y^j) \circ (\chi + X)(\beta(t) \cdot g)]$$

$$= [((\chi + X)_t)_\star \circ (\chi + X)_{-t})_\star \circ \sum_{j=1}^{d} u_j(\chi^j + Y^j) \circ (\chi + X)](\beta(t) \cdot g)$$

$$= [\sum_{j=1}^{d} u_j(\chi^j + Y^j) \circ (\chi + X)](\beta(t) \cdot g),$$

where, $(\ )_\star$ denotes the derivative. Thus, we get the form of the dynamic. □

Corollary 3.5. *Let Σ be an affine control system on a connected Lie group G with an observation (output) function h which is a Lie group homomorphism. Then, by the special form of the solution $\gamma(t)$, local and global observabilities of the system Σ depend on the drift vector field $(\chi + X)$.*

Proof. Each Z^i in the dynamic $\mathcal{D}$ induces a one-parameter group $Z^i_{t_i}$ of diffeomorphisms defined on G, and for each $i = 1, 2, \ldots, d$, there exists a differentiable curve $\beta_i : \mathbb{R} \to G$ such that for each $g \in G$,

$$Z^i_{t_i}(g) = (\chi + X)_{t_i}(\beta_i(t) \cdot g),$$

where $Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k} \in G_\Sigma$.

Since h is a homomorphism, $\forall g, l \in G$, we have the following

$$g \sim l \iff h(Z^i_{t_i}(g)) = h(Z^i_{t_i}(l)), \forall Z^i_{t_i} i \in S_\Sigma(g).$$

Thus

$$\begin{aligned}
&\iff h((\chi + X)_{t_i}(\beta_i(t) \cdot g)) = h((\chi + X)_{t_i}(\beta_i(t) \cdot l)), \forall t_i \geq 0;\\
&\iff h((\chi + X)_{t_i}(g)(\beta_i(t) \cdot e)) = h((\chi + X)_{t_i}(l)(\beta_i(t) \cdot e)), \forall t_i \geq 0;\\
&\iff h(\chi + X)_{t_i}(g) = h(\chi + X)_{t_i}(l), \forall t_i \geq 0;
\end{aligned}$$

where e denotes the identity element of G. □

Moreover, we know that the vector fields of the control system on a Lie group G are complete and therefore indistinguishability is an equivalence relationship. Hence, the set of indistinguishable elements from e is

$$\tilde{e} = I_e = \{g \in G | g \sim e\}$$

and for each $g \in G$, $I_g = \tilde{g} = g\tilde{e} = gI_e$. Indeed,

$$g \sim l \iff l \in gI_e.$$

Definition 3.6. *$\Sigma = (G, \mathcal{D}, h, V)$ be a control system on a connected Lie group G with observation (output). A nonempty subset $\hat{B} \subset G$ is called an observable set of the control system Σ if*

i) *it is S_Σ-invariant, i.e.*

$$h \circ S_\Sigma(g) \neq h \circ S_\Sigma(l) \iff g \nsim l, \forall g, l \in B$$

ii)

$$\hat{B} = G - \tilde{e}$$

iii) *maximal with respect to i) and ii).*

Remark 3.7. $\tilde{e}$ is an equivalence class and therefore it is a closed subset of Lie group G. Hence, $\hat{B}$ is open and has a differentiable manifold structure induced by G, [11].

In order to characterize observability of control systems on Lie groups, classical approach is to study the set of indistinguishable elements of the state space and then to find conditions for local and global observability, [3], [4] and [5]. In our work, we focus on the complement of the set of indistinguishable elements of the state spaces, i.e. our observable sets.

4. Existence of Observable Sets for Linear Control Systems on $\mathbb{R}^n$

A linear control system on $\mathbb{R}^n$ is a four-tuple denoted by $\Sigma = (\mathbb{R}^n, D, h, \mathbb{R}^s)$ and their dynamic is determined by the following differential equations:

$$\dot{x} = Ax + Bu$$
$$h(x) = Cx,$$

where $x \in \mathbb{R}^n$, A, B and C are matrices of appropriate orders and $h : \mathbb{R}^n \to \mathbb{R}^s$ is a linear function. Linear control systems on $\mathbb{R}^n$ are well-known and are important from their application points of view.

For any initial condition $x = x_0$, solution of the system has the following form:

$$\gamma(x_0, t, u) = e^{tA}\{x_0 + \int_0^t e^{-\tau A} Bu(\tau) d\tau\}.$$

In particular, if we consider the solution at the origin (i.e, identity element), then we get

$$\gamma(0, t, u) = e^{tA}\{\int_0^t e^{-\tau A} Bu(\tau) d\tau\}.$$

Thus, $S_\Sigma(0) = \{\gamma(0, t, u) : u \in \mathcal{U}, t \geq 0\}$ and the indistinguishability relation can be written by

$$x_1 \sim x_2 \iff C(e^{tA}(x_1)) = C(e^{tA}(x_2)),\ \forall t \geq 0$$

which is independent of the controlled vector field B.

Moreover, indistinguishable elements of this kind of systems from the identity element are characterized by

$$\tilde{0} = \bigcap_{i=0}^{n-1} Ker(CA^i).$$

Example 4.1. *We consider the following control system which is evolving according to Newton's law with the observation (output) function C given in [10]:*

$$\dot{x} = y$$
$$\dot{y} = u$$
$$h(x, y) = C(x, y)^T.$$

For any initial condition $(x, y) = (x_0, y_0)$, solution of the system has the following form:

$$\gamma((x_0, y_0), t, u) = (u\frac{t^2}{2} + y_0 t + x_0, ut + y_0)$$

and, in particular, we have the following at the identity:

$$\gamma((0, 0), t, u) = (u\frac{t^2}{2}, ut).$$

This system is globally observable, if for any two distinct points (x_1, y_1) and (x_2, y_2) of $\mathbb{R}^2$,

$$C(u\frac{t^2}{2} + y_1 t + x_1, ut + y_1) \neq C(u\frac{t^2}{2} + y_2 t + x_2, ut + y_2).$$

(a) *If we observe the position of the system, i.e., the output map is $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$, then by the observability rank condition this system is globally observable. Thus, observable set is the whole state space.*

Indeed, for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $(x_1, y_1) \neq (x_2, y_2)$, we have

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} u\frac{t^2}{2} + y_1 t + x_1 \\ ut + y_1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} u\frac{t^2}{2} + y_2 t + x_2 \\ ut + y_2 \end{pmatrix}.$$

(*b*) *If we observe the velocity of the system, i.e., the output map is* $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, *then by the observability rank condition this system is not globally observable.*

Indeed, $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ *and* $(x_1, y_1) \neq (x_2, y_2)$, *we have*

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} u\frac{t^2}{2} + y_1 t + x_1 \\ ut + y_1 \end{pmatrix} = ut + y_1$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} u\frac{t^2}{2} + y_2 t + x_2 \\ ut + y_2 \end{pmatrix} = ut + y_2.$$

Here, we do not have any information of the position of the system in the output space even if there is information on the velocity of the system.

Example 4.2. *Consider* $\Sigma = (\mathbb{R}^3, D, h, \mathbb{R}^2)$, *where*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then, the rank of $\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = 2 \neq 3$. *Therefore, this system is not globally observable.*

The set of indistinguishable elements from the neutral element is

$$\tilde{0} = \{(0, 0, k) | k \in \mathbb{R}\}$$

and the observable set is

$$\hat{B} = \mathbb{R}^3 - \{(0, 0, k) | k \in \mathbb{R}\}.$$

Existence of observable sets depends on the choices of observation (output) functions first and drift vector fields after. One can work on the same system by changing output functions or drift vector fields. Therefore, existence and optimization of the observable set is up to both output function and drift vector field.

5. Existence of Observable Sets for 3-dimensional Linear Control Systems on Heisenberg Lie groups

In this section, we consider a linear control system on a Heisenberg Lie group of dimension 3. A linear control system on a connected Lie group G is a four-tuple denoted by $\Sigma = (G, D, h, V)$ and their dynamic is determined by the following differential equations:

$$\dot{g}(t) = \chi^i(g(t)) + \sum_{j=1}^{d} Y^j(g(t))u(t)$$

$$h(g) = v \in V,$$

where $g \in G$, $\chi^1, \chi^2, \ldots, \chi^n$ are infinitesimal automorphisms, i.e.; each χ^i_t induces a 1-parameter group of automorphisms on G for each $t \in \mathbb{R}$, $Y^1, \ldots, Y^d \in L(G)$, the Lie algebra of G, and $h : G \to V$ is a differentiable function.

In [3], authors have given an algorithm where one can calculate indistinguishable points of linear controls systems on connected Lie groups with Lie group homomorphisms as observation (output) functions. We use this algorithm for 3-dimensional linear control systems on a Heisenberg Lie group in the example below and want to give its brief explanation. In the first and the second steps of the algorithm, the kernel of the observation (output) function and the basis of the Lie algebra of the kernel of observation (output) function are calculated, respectively. Later, the dual basis of the basis of the Lie algebra of the kernel of observation (output) function is found to determine dual of all indistinguishable points.

Example 5.1. *Consider Heisenberg Lie group of dimension 3*

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

and its Lie algebra is

$$L(H) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\},$$

where

$$span\left\{ X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$[X, Y] = Z.$$

Here,

$$[X, Y] = XY - YX.$$

(a) *We consider the following linear control system on H with observation (output) function p_d,*

$$\begin{cases} \Sigma = (H, \mathcal{D}, \pi, H/exp(\mathbb{R}Z)) \\ \\ p_d : H \to H/exp(\mathbb{R}Z) \end{cases}$$

Then, the kernel of the observation (output) function is

$$Ker(p_d) = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

and the basis of the Lie algebra of $Ker(p_d)$ is $\mathcal{B}_{K_p} = Z$. Besides, the dual is

$$\mathcal{B}_{K_p}^{\perp} = \{X^{\perp}, Y^{\perp}\}.$$

The adjoint action of the Lie algebra element X on Y, $ad_X(Y)$, is equal to $[X, Y] = Z$. Then, $ad_X(X) = ad_X(Z) = 0$. For the ordered basis (X, Y, Z) for ad_X, we have the following matrix

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It follows that, the co-adjoint actions of X, L_X, on $X^\perp$ and $Y^\perp$ are

$$L_X(X^\perp) = X^\perp R = 0$$
$$L_X(Y^\perp) = Y^\perp R = 0$$

and then the co-adjoint action of X, L_X, on $\mathcal{B}_{K_p}^\perp$ gives the following set

$$ad(X)\mathcal{B}_{K_p}^\perp = \{X^\perp, Y^\perp\} = \mathcal{I}^\perp.$$
$$\mathcal{I} = span\{Z\}.$$

Here, $\mathcal{I}$ is the Lie algebra of the Lie group I, the set of indistinguishable elements from the neutral element of H. Thus, Σ is not globally observable and the observable set $\hat{B} = H - exp(tZ)$.

(b) We consider the following linear control system on H with observation (output) function π, [3]

$$\begin{cases} \Sigma = (H, \mathcal{D}, \pi, H/exp(\mathbb{R}Y)) \\ \\ \quad \pi : H \to H/exp(\mathbb{R}Y) \end{cases}$$

Then, the kernel of the observation (output) function is

$$Ker(\pi) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

and the basis of the Lie algebra of $Ker(\pi)$ is $\mathcal{B}_K = Y$. Besides, the dual is

$$\mathcal{B}_K^\perp = \{X^\perp, Z^\perp\}.$$

It follows that, the co-adjoint action of X, L_X, on $Z^\perp$ is

$$L_X(Z^\perp) = Z^\perp R = Y^\perp$$

and then the co-adjoint action of X, L_X, on each element of $\mathcal{B}_K^\perp$ gives the following set

$$ad(X)\mathcal{B}_K^\perp = \{X^\perp, Y^\perp, Z^\perp\} = \mathcal{I}^\perp.$$

Thus,

$$\mathcal{I} = 0.$$

Thus, Σ *is locally observable, since the Lie algebra of the all indistinguishable elements is null. This system is also globally observable as given in [3]. Thus, the observable set* $\hat{B} = H$.

Conclusion:

In this work, we introduced the concept of observable sets. For this aim, we reviewed general control systems on Lie groups and gave the solution of the affine control system on a connected Lie group which represents a wide range of control systems on Lie groups. In the last two sections, we studied existence of observable sets for linear control systems on examples.

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Resumen

En este artículo, presentamos el concepto de conjuntos observables para sistemas de control. Por esta razón, revisamos sistemas de control generales en grupos de Lie con funciones de observación y damos la solución del sistema de control afín en un grupo de Lie conectado. Luego, estudiamos la existencia de conjuntos observables para sistemas de control lineal en ejemplos.

Palabras clave: conjunto observable, observabilidad, sistemas de control afín, sistemas de control lineal.

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THE GROEBNER BASIS AND SOLUTION SET OF A POLYNOMIAL SYSTEM RELATED TO THE JACOBIAN CONJECTURE[a]

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(*Presentado por R. Rabanal*)

Abstract

We compute the Groebner basis of a system of polynomial equations related to the Jacobian conjecture, and describe completely the solution set.

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Keywords: *Jacobian conjecture, Groebner basis*

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1. Introduction

Let K be a field of characteristic zero. The two–dimensional Jacobian Conjecture (JC), formulated by Keller in [7], asserts that any pair of polynomials $P, Q \in R := K[x, y]$ with

$$[P, Q] := \partial_x P \, \partial_y Q - \partial_x Q \, \partial_y P \in K^{\times}$$

defines an automorphism of R.

In [8], T. T. Moh investigated possible counterexamples (P, Q) of total degree below 101, identifying four exceptional pairs $(m, n) = (48, 64)$, $(50, 75)$, $(56, 84)$ and $(66, 99)$, where $(n, m) = (\deg P, \deg Q)$. He then ruled out these cases by explicitly solving certain ad-hoc systems of equations for the coefficients of the potential counterexamples.

Motivated by Moh's approach, in [5] the authors introduce a family of polynomial systems

$$\mathrm{St}(n, m, (\lambda_i), F_{1-n})$$

consisting of $m+n-2$ equations in $m+n-2$ variables with coefficients in a commutative K-algebra D. Here $(\lambda_i)_{0 \leq i \leq m+n-2} \subset K$ and $F_{1-n} \in D$. Among other results, they prove that a specific instance of this system (with $D = K[y]$ and $F_{1-n} = y$) has a solution in D^{m+n-2} if and only if there exists a counterexample (P, Q) to JC with $(n, m) = (\deg P, \deg Q)$. The argument relies on an equivalent formulation of JC due to Abhyankar in [1], which states that JC holds provided that for every Jacobian pair (P, Q) either $\deg P \mid \deg Q$ or $\deg Q \mid \deg P$. They also show that, when D is an integral domain, the set of solutions of $\mathrm{St}(n, m, (\lambda_i), F_{1-n})$ is finite. Furthermore, they examine in detail the "homogeneous" case $\lambda_i = 0$ for $i > 0$, giving an explicit description of its solutions. The usefulness of this method is shown in the last section of [5], where the method is illustrated with the case $(n, m) = (50, 75)$, showing—via a degree-reduction technique as in [4]—that no counterexample arises.

At the moment, there is no other method to discard small possible counterexamples arising from the lists of families of possible counterexamples given in [6] (see also [4]).

An advantage of this formulation is that the system of equations remains canonical, even under the modifications needed for computations as in [5]. This feature makes it suitable for algorithmic implementation and, potentially, for discarding infinite families of possible counterexamples rather than isolated cases.

In order to understand better the system, it could be helpful to understand a Groebner basis of the system. In [9], an explicit Groebner basis for the system $\mathrm{St}(2, m, (0), F_{1-n})$ is found. In the present paper we will analyze the system $\mathrm{St}(3, m, (0), F_{1-n})$. We first find a Groebner basis for a partial system, and then we manage to give a detailed description of the solution set.

2. The Jacobian conjecture as a system of equations

Let K be a characteristic zero field and let $K[y]((x^{-1}))$ be the algebra of Laurent series in x^{-1} with coefficients in $K[y]$. We will start from the following theorem, proved in [5, Theorem 1.9].

Theorem 2.1. *The Jacobian conjecture in dimension two is false if and only if there exist*

- $P, Q \in K[x, y]$ *and* $C, F \in K[y]((x^{-1}))$,
- $n, m \in \mathbb{N}$ *such that* $n \nmid m$ *and* $m \nmid n$,
- $\nu_i \in K$ *(*$i = 0, \dots, m+n-2$*) with* $\nu_0 = 1$,

such that

- C *has the form*

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \qquad \text{with each } C_{-i} \in K[y],$$

- *$gr(C) = 1$ and $gr(F) = 2 - n$, where gr is the total degree,*
- *$F_+ = x^{1-n}y$, where F_+ is the term of maximal degree in x of F,*
- *$C^n = P$ and $Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F$.*

Furthermore, under these conditions (P, Q) is a counterexample to the Jacobian conjecture.

In [5], the authors consider the following slightly more general situation. Let D be a K-algebra (for example, in Theorem 2.1 we have $D = K[y]$), n, m positive integers such that $n \nmid m$ and $m \nmid n$, $(\nu_i)_{1\leq i\leq n+m-2}$ a family of elements in K with $\nu_0 = 1$ and $F_{1-n} \in D$ (in Theorem 2.1 we take $F_{1-n} = y$). A Laurent series in x^{-1} of the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \qquad \text{with } C_{-i} \in D,$$

is a *solution of the system* $S(n, m, (\nu_i), F_{1-n})$, if there exist $P, Q \in D[x]$ and $F \in D[[x^{-1}]]$, such that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + \cdots, \quad \text{with } F_{1-n}, F_{-n}, \ldots \text{ in } D,$$

$$P = C^n \qquad \text{and} \qquad Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F.$$

For example, if $n = 3$, then

$$\begin{aligned} P(\mathbf{x}) = C^3 = {} & \mathbf{x}^3 + 3C_{-1}\ \mathbf{x} + 3C_{-2} + (3C_{-1}^2 + 3C_{-3})\ \mathbf{x}^{-1} \\ & + (6C_{-1}C_{-2} + 4C_{-4})\ \mathbf{x}^{-2} \\ & + (C_{-1}^3 + 3C_{-2}^2 + 6c_{-2}C_{-3} + 3C_{-5})\ \mathbf{x}^{-3} \\ & + (3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6})\ \mathbf{x}^{-4} \\ & + (3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} \\ & \quad + 6C_{-7})\ \mathbf{x}^{-5} \\ & + \ldots \end{aligned}$$

and the condition $C^3 \in K[x]$ translates into the following conditions on C_{-k}:

$$
\begin{aligned}
0 &= (C^3)_{-1} = 3C_{-1}^2 + 3C_{-3},\\
0 &= (C^3)_{-2} = 6C_{-1}C_{-2} + 4C_{-4},\\
0 &= (C^3)_{-3} = C_{-1}^3 + 3C_{-2}^2 + 6c_{-2}C_{-3} + 3C_{-5},\\
0 &= (C^3)_{-4} = 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6},\\
0 &= (C^3)_{-5} = 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5}\\
&\qquad + 6C_{-7},\\
&\quad\vdots
\end{aligned}
$$

In the general case, the condition $P(x) = C^n \in K[x]$ yields the equations $(C^n)_{-k} = 0$, whereas the condition $Q(x) = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F \in K[x]$ gives us the equations $\left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F\right)_{-k} = 0$, where we note that $F_{-k} = 0$ for $k = 1, \dots, n-2$.

It is easy to see (e.g. [5, Remark 1.13]) that the first $m+n-2$ coefficients determine the others, i.e., the coefficients $C_{-1}, \dots, C_{-m-n+2}$ determine univocally the coefficients C_{-k} for $k > m+n-2$. Moreover, the F_{-k} for $k > n-1$ depend only on F_{1-n} and C. Consequently, having a solution C to the system $S(n, m, (\nu_i), F_{1-n})$ is the same as having a solution $(C_{-1}, \dots, C_{-m-n+2})$ to the system

$$
\begin{aligned}
E_k &:= (C^n)_{-k} = 0, && \text{for } k = 1, \dots, m-1,\\
E_{m-1+k} &:= \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i}\right)_{-k} = 0, && \text{for } k = 1, \dots, n-2,\\
E_{m+n-2} &:= \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i}\right)_{1-n} + F_{1-n} = 0,
\end{aligned}
\tag{2.1}
$$

with $m+n-2$ equations $E_k = 0$ and $m+n-2$ unknowns C_{-k}.

In order to understand the solution set of this system, it would be very helpful to find a Groebner basis for the ideal generated by the polynomials E_k in $D[C_{-1},\dots,C_{m+n-2}]$. In this paper we compute such a Groebner basis of (2.1) in a very particular case: we assume $n=3$, $m=3r+1$ or $m=3r+2$ for some integer $r>0$, and $\nu_i=0$ for $i>0$. Moreover we consider $D=\mathbb{C}[y]$ and $F_{1-n}=y$, as in Theorem 2.1.

3. Computation of a Groebner basis for I_{m-1}

Assume $n=3$, $3\nmid m>3$ and $\nu_i=0$ for $i>0$. Set also $D=\mathbb{C}[y]$ and $F_{1-n}=y$.

Then the system (2.1) reads

$$E_i = \begin{cases} (C^3)_{-i}, & i=1,\dots,m-1,\\ (C^m)_{-1}, & i=m,\\ (C^m)_{-2}+y, & i=m+1, \end{cases} \tag{3.1}$$

where $(C^2)_{-i}$ denotes the coefficient of x^{-i} in the Laurent series C^3. Explicitly, the polynomials E_i are given by

$$\begin{aligned} E_1 &= 3C_{-1}^2+3C_{-3},\\ E_2 &= 6C_{-1}C_{-2}+3C_{-4},\\ E_3 &= C_{-1}^3+3C_{-2}^2+6C_{-1}C_{-3}+3C_{-5},\\ E_4 &= 3C_{-1}^2C_{-2}+6C_{-2}C_{-3}+6C_{-1}C_{-4}+3C_{-6},\\ E_5 &= 3C_{-1}C_{-2}^2+3C_{-1}^2C_{-3}+3C_{-3}^2+6C_{-2}C_{-4}+6C_{-1}C_{-5}+3C_{-7},\\ &\vdots\\ E_{m-1} &= (C^3)_{1-m},\\ E_m &= (C^m)_{-1},\\ E_{m+1} &= (C^m)_{-2}+y. \end{aligned} \tag{3.2}$$

Each E_i is a polynomial in the ring $\mathbb{C}[C_{-1}, C_{-2}, \ldots, C_{m+1}, y]$, and the $m+1$ polynomials yield the ideal

$$I \quad = \quad \langle E_1, \ldots, E_m, E_{m+1} \rangle.$$

Our goal is to find a Groebner basis for the ideal I, but we find it nearly explicit only for $I_{m-1} := \langle E_1, E_2, \ldots, E_{m-2}, E_{m-1} \rangle$. For this we note that the equations are homogeneous, for the weight obtained by setting

$$w(C_{-i}) = i+1, \quad \text{and} \quad w(y) = m+n-1 = m+2.$$

We consider y as a variable, so the equations remain homogeneous. Then

$$w(E_k) = k{+}3, \text{ for } k = 1, \ldots, m-1, \quad w(E_m) = m{+}1 \quad w(E_{m+1}) = m{+}2.$$

Note that for $k = 1 \ldots, m-1$ we have

$$\begin{aligned} E_k :=& 3\Big(\sum_{\substack{i=-1 \\ 3i \neq k}}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)} \Big) + 6 \Big(\sum_{\substack{0<i<j \\ i+j=k+1}} C_{-i} C_{-j} \Big) \\ &+ 6 \Big(\sum_{\substack{0<i<j<l \\ i+j+l=k}} C_{-i} C_{-j} C_{-l} \Big) + \varepsilon (C_{-\frac{k}{3}})^3, \end{aligned} \tag{3.3}$$

where

$$\varepsilon = \begin{cases} 1, & 3|k \\ 0, & 3 \nmid k \end{cases} .$$

Note that $C_1 = 1$ and $C_0 = 0$, and so

$$3 \sum_{i=-1}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)} = 3C_{k+2} + 3 \sum_{i=1}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)}. \tag{3.4}$$

In order to compute a Groebner basis we will consider the degree reverse lexicographic monomial order, but for the degree given by the above mentioned weight. This means that the monomial order is given by the matrix

$$\text{wmat} = \begin{pmatrix} m+2 & m+1 & m & \dots & 4 & 3 & 2 & m+2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

on the variables $C_{-(m+1)}, C_{-m}, C_{-(m-1)}, \dots, C_{-3}, C_{-2}, C_{-1}, y$. We first compute the reduced Groebner basis $(\widetilde{E}_1, \widetilde{E}_2, \dots, \widetilde{E}_{m-1})$ for the ideal $I_{m-1} := \langle E_1, E_2, \dots, E_{m-2}, E_{m-1}\rangle$.

Proposition 3.1. *The set $\{E_1, \dots, E_{m-1}\}$ is a Groebner basis of I_{m-1}. The reduced Groebner basis of I_{m-1} is given by polynomials $\widetilde{E}_k$ for $k = 1, \dots, m-1$, each of the form*

$$\widetilde{E}_k = C_{-(k+2)} + R_k(C_{-1}, C_{-2}),$$

where $R_k(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ is an homogeneous polynomial in the variables C_{-1} and C_{-2} of weight $w(\widetilde{E}_k) = w(E_k) = k+3$.

Proof. By (3.3) and (3.4) we know that E_k is of the form

$$E_k = 3C_{-k-2} + T(C_{-1}, \dots, C_{-k}), \quad \text{for } k = 1, \dots, m-1,$$

where T is a polynomial in the variables $C_{-1}, \dots, C_{-k}$. Then by Proposition 2.9.4 of [2], since

$$\begin{aligned} LCM(LT(E_i)/3, LT(E_j)/3) &= LCM(C_{-i-2}, C_{-j-2}) = C_{-i-2}C_{-j-2} \\ &= (LT(E_i)/3)(LT(E_j)/3) \end{aligned}$$

we have $S(E_i, E_j) \longrightarrow_G 0$, and so, by Theorem 2.9.3 of [2], the set $G = \{E_1/3, \ldots, E_{m-1}/3\}$ is a Groebner basis of I_{m-1}. One verifies directly that it is a minimal Groebner basis, according to Definition 2.7.4 of [2]. If we apply the process described in the proof of [2, Proposition 2.7.6] to the Groebner basis $G = \{E_1/3, \ldots, E_{m-1}/3\}$ we obtain that

$$\widetilde{E}_1 = \overline{E_1/3}^{G\setminus\{\frac{E_1}{3}\}} = E_1/3 \quad \text{and} \quad \widetilde{E}_1 = \overline{E_2/3}^{G\setminus\{\frac{E_2}{3}\}} = E_2/3.$$

Moreover, for $k = 3, \ldots, m-1$, set $G_k = \{\widetilde{E_1}, \ldots, \widetilde{E_{k-1}}, E_k, \ldots, E_{m-1}\}$ and then

$$\widetilde{E}_k = \overline{E_k}^{G_k \setminus E_k}.$$

Clearly the remainder can have only the variables C_{-1} and C_{-2}, hence $\widetilde{E}_k$ is of the form

$$\widetilde{E}_k = C_{-(k+2)} + R_k(C_{-1}, C_{-2}),$$

as desired. □

Although we have no explicit formula for $R_k(C_{-1}, C_{-2})$, we can compute it for small k.

$$\begin{aligned} \tilde{E}_1 &= C_{-3} + C_{-1}^2, \\ \tilde{E}_2 &= C_{-4} + 2C_{-1}C_{-2}, \\ \tilde{E}_3 &= C_{-5} + C_{-2}^2 - \frac{5}{3}C_{-1}^3, \\ \tilde{E}_4 &= C_{-6} - 5C_{-1}^2C_{-2}, \\ \tilde{E}_5 &= C_{-7} + \frac{10}{3}C_{-1}^4 - 5C_{-1}C_{-2}^2. \end{aligned}$$

Dividing the polynomials E_m and E_{m+1} by the polynomials

$$\{\widetilde{E}_{m-1}, \ldots, \widetilde{E}_2, \widetilde{E}_1\}$$

with respect to the given order, we obtain

$$\overline{\frac{E_m}{3}}^{G_m \setminus \{\frac{E_m}{3}\}} = \widetilde{E}_m = R_m(C_{-1}, C_{-2})$$

and

$$\overline{\frac{E_{m+1}}{3}}^{G_m \setminus \{\frac{E_{m+1}}{3}\}} = \widetilde{E}_{m+1} = y + R_{m+1}(C_{-1}, C_{-2}),$$

where $R_m(C_{-1}, C_{-2}), R_{m+1}(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ are homogeneous polynomials such that $w(\widetilde{E}_m) = w(E_m) = m + 1$ and $w(\widetilde{E}_{m+1}) = w(E_{m+1}) = m + 2$.

Although we don't give an explicit description of the Groebner Basis of the whole system, in the next section we show how to determine the solution set of the polynomial system, using that

$$I = \langle E_1, E_2, \ldots, E_m, E_{m+1}\rangle = \langle \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_m, \widetilde{E}_{m+1}\rangle.$$

4. The solution set of the system of polynomial equations

In this section we analyze the solutions of the system of equations. Note that the partial system I_{m-1} shows that the values of C_{-1} and C_{-2} determine univocally the values of C_{-k} for $k > 2$. Moreover, C_{-1} and C_{-2} can be computed using the following two equations:

$$\widetilde{E}_m = R_m(C_{-1}, C_{-2}) = 0 \tag{4.1}$$

and

$$\widetilde{E}_{m+1} = y + R_{m+1}(C_{-1}, C_{-2}) = 0, \tag{4.2}$$

where $R_m(C_{-1}, C_{-2}), R_{m+1}(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ are homogeneous polynomials with respect to the weight considered before, i.e. $w(C_{-1}) =$

2, $w(C_{-2}) = 3$. Moreover $w(\widetilde{E}_m) = m+n-2 = m+1$ and $w(\widetilde{E}_{m+1}) = m+n-1 = m+2$. Then (4.1) and (4.2) read

$$\widetilde{E}_m = \sum_{2i+3j=m+1} \lambda_m^{ij} C_{-1}^i C_{-2}^j \tag{4.3}$$

and

$$\widetilde{E}_{m+1} = y + \sum_{2i+3j=m+2} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j, \tag{4.4}$$

for some constants $\lambda_m^{ij}, \lambda_{m+1}^{ij} \in K$. By (4.4) the two variables cannot be zero at the same time. We compute first the solutions in the cases where one of the variables is zero.

FIRST CASE: $C_{-1} = 0$ and $C_{-2} \neq 0$.

In this case the only term surviving in (4.3) is

$$0 = \widetilde{E}_m = \lambda_m^{0j} C_{-2}^j,$$

with $3j = m+1$. So necessarily

$$\lambda_m^{0,(m+1)/3} = 0 \quad \text{if} \quad 3|m+1. \tag{4.5}$$

Similarly, the only term surviving in the sum (4.4) has $i = 0$, and so we obtain

$$0 = \widetilde{E}_{m+1} = y + \lambda_{m+1}^{0j} C_{-2}^j \quad \text{with } 3j = m+2.$$

Since $y \neq 0$, necessarily $\lambda_{m+1}^{0j} \neq 0$ for $3j = m+2$, and so $3|m+2$, i.e. $m \equiv 1 \mod 3$. This shows that the condition (4.7) is trivially satisfied.

Lemma 4.1. *If* $3|m+2$, *and* $C_{-1} = 0$, *then* $\lambda_{m+1}^{0j} \neq 0$ *for* $3j = m+2$.

Proof. It is easy to check that $P = x^3 + 3C_{-2}$, and then, by Newtons binomial theorem we have

$$C^m = P^{m/3} = \sum_{k=0}^{\infty} \binom{m/3}{k} (3C_{-2})^k (x^3)^{\frac{m}{3}-k}. \tag{4.6}$$

Thus $\lambda^{0j}_{m+1}C^j_{-2} = (C^m)_{-2}$ is the coefficient of $x^{-2} = (x^3)^{\frac{m}{3}-j}$, since $m = 3j - 2$. Then

$$\lambda^{0j}_{m+1} = \binom{m/3}{j} 3^j \neq 0,$$

as desired. □

Thus we have proved the following proposition.

Proposition 4.2. *If* $(C_{-1}, C_{-2}, \ldots, C_{-(m+1)})$ *is a solution of the system* (3.1)*, with* $C_{-1} = 0$ *and* $C_{-2} \neq 0$*, then*

- $m \equiv 1 \mod 3$*,*
- $\lambda^{0j}_{m+1} \neq 0$ *for* $j := \frac{m+2}{3}$*,*
- *There are* j *solutions of the system* (3.1) *in* $K[y^{1/j}]$*, given by*

$$C_{-1} = 0, \quad C_{-2} = \left(\frac{-y}{\lambda^{0j}_{m+1}}\right)^{\frac{1}{j}} \quad \text{and} \quad C_{-k} = -R_{k-2}(C_{-1}, C_{-2})$$

for $3 \leq k \leq m + 1$*.*

SECOND CASE: $C_{-1} \neq 0$ and $C_{-2} = 0$.

In this case the only term surviving in (4.3) is

$$0 = \widetilde{E}_m = \lambda^{i0}_m C^i_{-1},$$

with $2i = m + 1$. So necessarily

$$\lambda^{(m+1)/2,0}_m = 0 \quad \text{if} \quad 2|m + 1. \tag{4.7}$$

Similarly, the only term surviving in the sum (4.4) has $j = 0$, and so we obtain

$$0 = \widetilde{E}_{m+1} = y + \lambda^{i0}_{m+1} C^i_{-1} \quad \text{with } 2i = m + 2.$$

Since $y \neq 0$, necessarily $\lambda^{i0}_{m+1} \neq 0$ for $2i = m + 2$, and so $2|m + 2$, i.e. m is even. This shows that the condition (4.7) is trivially satisfied.

Lemma 4.3. *If $2|m$ and $C_{-2}=0$, then $\lambda^{i0}_{m+1}\neq 0$ for $2i=m+2$.*

Proof. It is easy to check that $P=x^3+3xC_{-1}$, and then, by Newtons binomial theorem we have

$$C^m = P^{m/3} = \sum_{k=0}^{\infty}\binom{m/3}{k}(3xC_{-1})^k(x^3)^{\frac{m}{3}-k}. \tag{4.8}$$

Thus $\lambda^{i0}_{m+1}C^i_{-1}=(C^m)_{-2}$ is the coefficient of $x^{-2}=(x)^i(x^3)^{\frac{m}{3}-i}$, since $m=2i-2$. Then

$$\lambda^{i0}_{m+1}=\binom{m/3}{i}3^i\neq 0,$$

as desired. □

Thus we have proved the following proposition.

Proposition 4.4. *If $(C_{-1},C_{-2},\dots,C_{-(m+1)})$ is a solution of the system (3.1), with $C_{-1}\neq 0$ and $C_{-2}=0$, then*

- $m\equiv 1 \mod 3$,
- $\lambda^{i0}_{m+1}\neq 0$ *for* $i:=\frac{m+2}{2}$,
- *There are i solutions of the system (3.1) in $K[y^{1/i}]$, given by*

  $$C_{-1}=\left(\frac{-y}{\lambda^{i0}_{m+1}}\right)^{\frac{1}{i}},\quad C_{-2}=0 \quad \textit{and} \quad C_{-k}=-R_{k-2}(C_{-1},C_{-2})$$

  for $3\leq k\leq m+1$.

THIRD CASE: $C_{-1}\neq 0$, $C_{-2}\neq 0$ and m even.
In this case we introduce a new auxiliary variable t satisfying $C^2_{-2}=tC^3_{-1}$. The equality (4.3) now reads

$$\begin{aligned}\widetilde{E}_m &= \sum_{2i+3j=m+1}\lambda^{ij}_m C^i_{-1}C^j_{-2} = \sum_{2i+6r+3=m+1}\lambda^{i,2r+1}_m C^i_{-1}C^{2r+1}_{-2}\\ &= \sum_{2i+6r+2=m}\lambda^{i,2r+1}_m C^{i+3r}_{-1}C_{-2}t^r,\end{aligned}$$

since m even implies that the weight $2i+3j=m+1$ is odd, so j is odd and can be written as $2r+1$. Moreover, for the terms in the sum we have $i+3r=\frac{m-2}{2}$, and so we arrive at

$$0=C_{-1}^{\frac{m-2}{2}}C_{-2}\sum_{\substack{2i+6r=m-2\\ j=2r+1}}\lambda_m^{ij}t^r.$$

Thus t is a root of the polynomial

$$f(t)=\sum_{r=0}^{\lfloor\frac{m-2}{6}\rfloor}a_r t^r,\quad \text{where } a_r=\lambda_m^{\frac{m-2-6r}{2},2r+1}. \tag{4.9}$$

Let $\{t_1,\ldots,t_s\}$ be the roots of the polynomial $f(t)$. Note that in the equality (4.4) the power j has to be even, since m is even and $2i+3j=m+2$. Hence, if we replace C_{-2}^2 by $t_l C_{-1}^3$ in (4.4), we obtain

$$\widetilde{E}_{m+1}=y+\sum_{\substack{2i+3j=m+2\\ j=2r}}\lambda_{m+1}^{ij}C_{-1}^iC_{-2}^j=y+\sum_{2i+6r=m+2}\lambda_{m+1}^{i,2r}C_{-1}^{i+3r}t_l^r.$$

Note that for each of the terms in the last sum we have $i+3r=\frac{m+2}{2}$, and so

$$0=y+C_{-1}^{\frac{m+2}{2}}g(t_l),\quad \text{where } g(t)=\sum_{r=0}^{\lfloor\frac{m+2}{6}\rfloor}b_r t^r,$$

with $b_r=\lambda_{m+1}^{\frac{m+2-6r}{2},2r}$. It follows that

$$C_{-1}=\left(\frac{-y}{g(t_l)}\right)^{\frac{2}{m+2}}.$$

Thus we have arrived at the following result.

Proposition 4.5. *If* $(C_{-1},C_{-2},\ldots,C_{-(m+1)})$ *is a solution of the system* (3.1)*, with* $C_{-1}\neq 0$*,* $C_{-2}\neq 0$ *and* m *even, then the system has at most* $s\cdot(m+2)$ *solutions, where* s *is the number of roots of*

$f(t)$ defined in (4.9). Moreover, for every choice of a root t_l of f, the solutions are given by

$$\begin{aligned} C_{-1} &= \left(\frac{-y}{g(t_l)}\right)^{\frac{2}{m+2}}, && \frac{m+2}{2} \quad \textit{choices},\\ C_{-2} &= \left(t_l C_{-1}^3\right)^{\frac{1}{2}}, && 2 \quad \textit{choices},\\ C_{-k} &= -R_{k-2}(C_{-1}, C_{-2}) \quad \textit{for} \quad 3 \le k \le m+1. \end{aligned}$$

FOURTH CASE: $C_{-1} \neq 0$, $C_{-2} \neq 0$ and m odd.
In this case we introduce a new auxiliary variable t satisfying $C_{-2}^2 = tC_{-1}^3$. The equality (4.3) now reads

$$\begin{aligned} \tilde{E}_m &= \sum_{2i+3j=m+1} \lambda_m^{ij} C_{-1}^i C_{-2}^j = \sum_{2i+6r=m+1} \lambda_m^{i,2r} C_{-1}^i C_{-2}^{2r}\\ &= \sum_{2i+6r=m+1} \lambda_m^{i,2r} C_{-1}^{i+3r} t^r, \end{aligned}$$

since m odd implies that the weight $2i+3j = m+1$ is even, so j is even and can be written as $2r$. Moreover, for the terms in the sum we have $i + 3r = \frac{m+1}{2}$, and so we arrive at

$$0 = C_{-1}^{\frac{m+1}{2}} \sum_{\substack{2i+6r=m+1\\ j=2r}} \lambda_m^{ij} t^r.$$

Thus t is a root of the polynomial

$$f(t) = \sum_{r=0}^{\lfloor \frac{m+1}{6} \rfloor} a_r t^r, \quad \text{where } a_r = \lambda_m^{\frac{m+1-6r}{2},2r}. \tag{4.10}$$

Let $\{t_1, \dots, t_s\}$ be the roots of the polynomial $f(t)$. Note that in the equality (4.4) the power j has to be odd, since m is odd and $2i+3j = m+2$. Hence, if we replace C_{-2}^2 by $t_l C_{-1}^3$ in (4.4), we obtain

$$\widetilde{E}_{m+1} = y + \sum_{\substack{2i+3j=m+2\\ j=2r+1}} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j = y + \sum_{2i+6r+3=m+2} \lambda_{m+1}^{i,2r+1} C_{-1}^{i+3r} C_{-2} t_l^r.$$

Note that for each of the terms in the last sum we have $i + 3r = \frac{m-1}{2}$, and so

$$0 = y + C_{-1}^{\frac{m-1}{2}} C_{-2} g(t_l), \quad \text{where } g(t) = \sum_{r=0}^{\lfloor \frac{m-1}{6} \rfloor} b_r t^r,$$

with $b_r = \lambda_{m+1}^{\frac{m-1-6r}{2}, 2r+1}$. We also replace C_{-2} by $\big(t_l C_{-1}^3\big)^{\frac{1}{2}}$. It follows that

$$0 = y + C_{-1}^{\frac{m+2}{2}} (t_l)^{\frac{1}{2}} g(t_l),$$

and so

$$C_{-1} = \Big(\frac{-y}{(t_l)^{\frac{1}{2}} g(t_l)}\Big)^{\frac{2}{m+2}}.$$

Thus we have arrived at the following result.

Proposition 4.6. *If* $\big(C_{-1}, C_{-2}, \ldots, C_{-(m+1)}\big)$ *is a solution of the system* (3.1)*, with* $C_{-1} \neq 0$*,* $C_{-2} \neq 0$ *and* m *odd, then the system has at most* $2 \cdot s \cdot (m+2)$ *solutions, where* s *is the number of roots of* $f(t)$ *defined in* (4.10)*. Moreover, for every choice of a root* t_l *of* f*, we first choose a square root of* t_l *and then the solutions are given by*

$$C_{-1} = \Big(\frac{-y}{(t_l)^{\frac{1}{2}} g(t_l)}\Big)^{\frac{2}{m+2}}, \qquad \frac{m+2}{2} \quad \textit{choices,}$$

$$C_{-2} = \big(t_l C_{-1}^3\big)^{\frac{1}{2}}, \qquad 2 \quad \textit{choices,}$$

$$C_{-k} = -R_{k-2}(C_{-1}, C_{-2}) \quad \textit{for} \quad 3 \leq k \leq m+1.$$

Final Remark.

The solution sets arising in the four cases reveal that no solution exists in $K[y]$, whereas all solutions lie in $K\big[y^{1/(m+2)}\big]$. This, in turn, implies that there is no counterexample (P, Q) to the Jacobian Conjecture with $\deg(P) = 3$ and $3 \nmid \deg(Q)$. Although this fact is already known—for instance, because no counterexample can occur when $\gcd\big(\deg(P), \deg(Q)\big) = 1$—a more detailed analysis of the corresponding Gröbner bases in broader settings may still yield new insights toward a proof or disproof of the Jacobian Conjecture.

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Resumen

Calculamos la base de Groebner de un sistema de ecuaciones polinomiales relacionadas con la conjetura jacobiana y describimos completamente el conjunto de soluciones.

Palabras clave: Conjetura jacobiana. Base de Groebner

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